Module - 4

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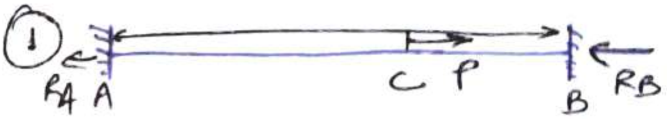
Matrix MTD of Structural Analysis :-

Concept 1 :→

1.) MTD to analysis Statically Indeterminate Structure

➢ All the MTD of analysis can be classified into two types, depending on the types of unknown chosen for the analysis {i.e., force MTD & displacement MTD}.

Force/Flexibility/Compatibility MTD	Displacement/Stiffeness/Equillibrium MTD
Characteristics	Characteristics :-
(i) Forces are taken as unknown {i.e., Reaction SF & BM} & equation are expressed in terms of these forces.	(i) Displacement {i.e., slope deflections} are taken as unknown and eq^n are expressed in term of these unknown displacement {Slope-deflection eq^n}.
(ii) Additional eq^n called compatibility condition are developed to find all the unknown forces.	(ii) Additional joint equilibrium eq^n are development to find find unknown displacement
ex:- In strength of Materials:-	ex:- ① Slope deflection MTD.
① 	② Moment distribution MTD. {Successive distribution MTD}
$D_s = r - S = 2 - 1 = 1$	③ Kani's MTD {Iterative version of SDM}.
So, 1 compatibility condition are reqd. to find all forces.	④ Stiffeness Matrix MTD {Matrix version of SdM}.
$\Sigma x = 0 \Rightarrow$ $-R_A - R_B + P = 0$	

Compatibility condition ⇒
elongation in AC = contraction in BC.

{A relationship b/w deformation}

2. In Flitched beams :-

Equi eqⁿ ⇒ $(M.R) = (M.R)_{steel} + (M.R)_{load}$ — (1)

Compatibility condition ⇒

$(R)_{steel} = (R)_{wood}$ — (2)

$(e)_{steel} = (e)_{wood}$.

(3) In structural analysis :-

(1) Theorem of 3-moment
{Compatibility condition ⇒
$\theta_{BA} = \theta_{BC}$}.

(2) Redundant trusses {by strain energy or unit load MTD}
⇒ Compatibility condition
⇒ $\left(\delta_{AC} = -\frac{xL}{AE}\right)$

(3) 2-hinge Arch ⇒ Compatibility Condition ⇒ $\frac{\partial U}{\partial H} = 0$

(4) Flexibility Matrix MTD

Note :- If Ds is less than this MTD is used

Note :- If Dk is less then these MTD are suitable.

→

Stiffness Matrix MTD of Analysis :-

Concept ①

Stiffness (K) :-

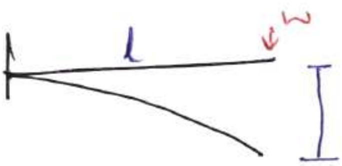
The load reqd. to produce unit deflection (displacement).

Ex:- ① Axial stiffness of a spring $= K = \frac{W}{\delta} = \frac{W}{\frac{64 W R^3 n}{C \cdot d^4}}$

$$= \frac{C \cdot d^4}{64 \cdot R^3 \cdot n}$$

(ii) Torsional stiffness of a spring $= K = T/\theta$

(iii) Flexural stiffness of a beam shown in fig is ?



$$\delta = \frac{Wl^3}{3EI}, \quad K = \frac{W}{\delta} = \frac{W}{\frac{Wl^3}{3EI}} = \frac{3EI}{l^3}$$

Concept ② Types of Stiffness :-

ⓐ Axial Stiffeness (K_{11}) :-

force at co-ordinate ① to produce unit displacement at co-ordinate ①

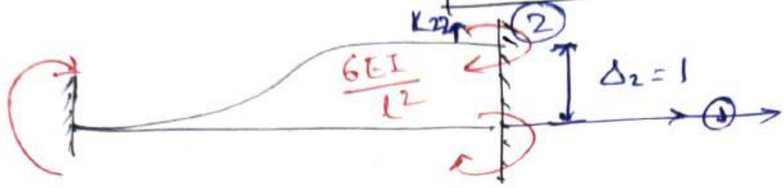
{load reqd. to produce unit elongation }.

$$= \frac{P_1}{\Delta_1} = \frac{P_1}{\frac{P_1 l}{AE}} = \boxed{\frac{AE}{l}}$$

(b) Transverse stiffness (K_{22}) :-

⇒ Force at co-ordinate ② to produce unit displacement at ② {By fixing all other co-ordinate displacement}.

$$K_{22} = \frac{12EI\Delta_2}{l^3} = \boxed{\frac{12EI}{l^3}}$$ ××

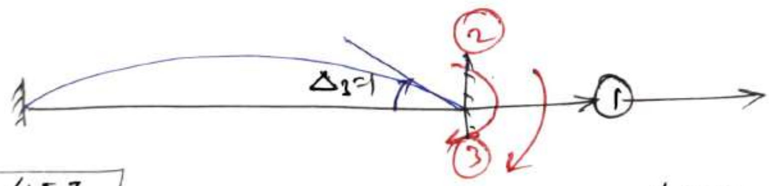


$$\frac{6EI\Delta_2}{l^2} = \frac{6EI}{l^2}$$

(c) Flexural stiffness (K_{33}) :-

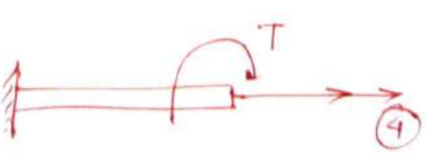
{Stiffness factor in M.D.M}

⇒ Force at ③ due to unit displacement at ③ {i.e., Moment reqd. to produce unit rotation}.



$$\boxed{K_{33} = \frac{4EI}{l}}$$

$$K_{33} = \frac{4EI}{l}$$



(d) Tortional stiffness (K_{44}) :-

Force at co-ordinate ④ due to unit displacement at co-ordinate ④ {Torque reqd. to produce unit twist}

$$\Rightarrow K_{44} = \frac{T}{\theta} = \frac{T}{\left(\frac{Tl}{CJ}\right)} = \boxed{\frac{CJ}{l}}$$

Concept (3) :-

Characteristics of stiffeness matrix [k] :-

(1) To get first column of stiffness matrix. fix all co-ordinate and applying unit displacement at co-ordinate (1) and find forces developed at all co-ordinates. Similarly to get second column of stiffness matrix. apply unit displacement at co-ordinate (2) and find forces developement at all co-ordinate.

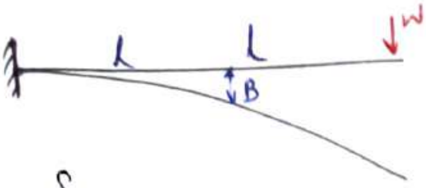
(11) Stiffeness matrix is a square symmetric matrix:-
{Symmetric bcz of maxwell's reciprocal deflection theorem} :-

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

K_{12} = force at (1) due to unit displacement at (2),
K_{21} = force at (2) due to unit displacement (1)

$K_{12} = K_{21}$

From Maxwell's reciprocal deflection theorem:-



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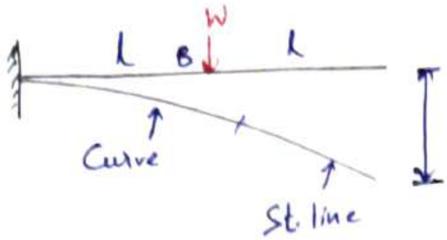


$\delta_B =$

$$\delta_c = \frac{wl^3}{3EI} + \frac{wl^2}{2EI}(2l - l)$$

$$\boxed{\delta_B = \delta_c = \frac{5wl^3}{6EI}}$$

$$\boxed{\delta_c = \frac{5wl^3}{6EI}}$$

(iii) Stiffness matrix diagonal elements are always (+ve)
{from Castigliano's theorem-2}.

(iv) Stiffness matrix can be developed only when the structure is stable.

(v) If displacement at any co-ordinate is impossible then, stiffness matrix for that co-ordinate system will not exist.

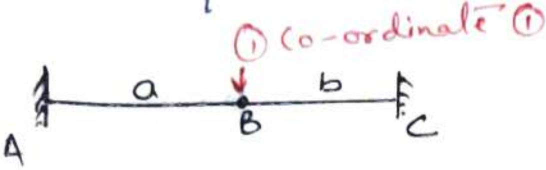
ex:-

Axially rigid member ⇒ $AE = \infty$

$$\boxed{K_{11} = \frac{P_1}{\Delta_1 (= 0)} = \infty}$$

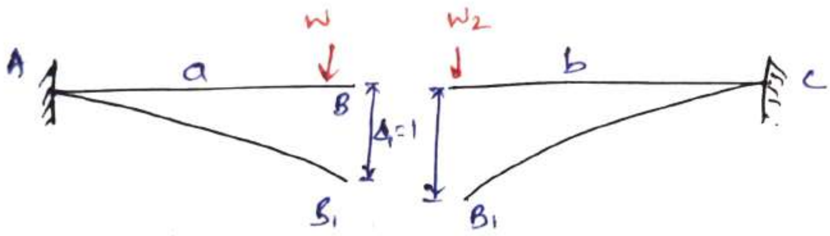
→

Q. Stiffeness co-efficient K_{11}, for the beam shown in fig. is ? {2 Marks}.



K_{11} = force at ① to produce unit displacement at ①

Sol^n:



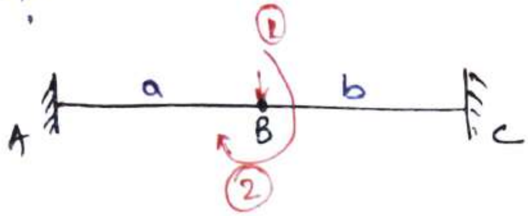
$$\Delta_1 = \frac{W_1 \cdot a^3}{3EI} \qquad \Delta_2 = \frac{W_2\, b^3}{3EI}$$

$$\Rightarrow W_1 = \frac{3EI}{a^3} \qquad \Rightarrow W_2 = \frac{3EI}{b^3}$$

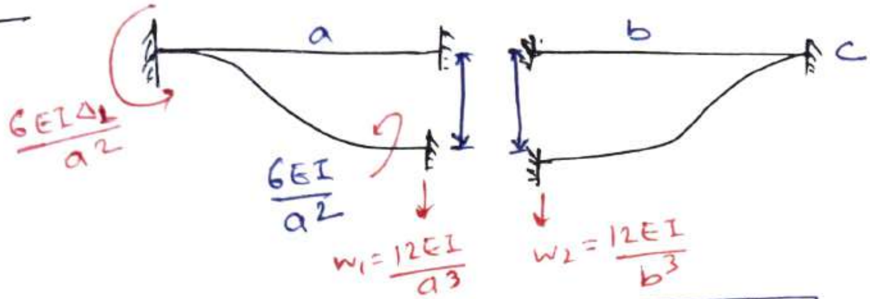
$$K_{11} = W_1 + W_2 = \frac{3EI}{a^3} + \frac{3EI}{b^3}.$$

$$\boxed{K_{11} = 3EI\left[\frac{1}{a^3} + \frac{1}{b^3}\right]}.$$

Q. For the structure shown in fig. stiffness coefficient k_{11} is?



Sol^n:



$$K_{11} = W_1 + W_2 = 12EI\left[\frac{1}{a^3} + \frac{1}{b^3}\right]$$

Conclusion :-

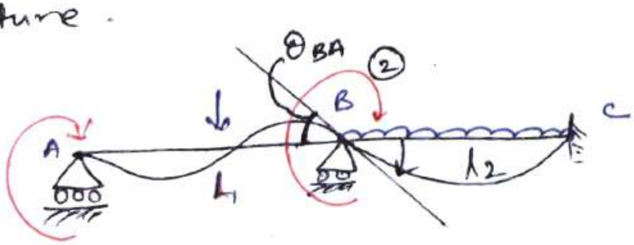
Depending on the co-ordinate system stiffeness co-efficient with be different.

➢ Procedure for stiffeness matrix MTD:-

Step1:- find Dk, give co-ordinate number to displacements.

fix all co-ordinate and get a restricted structure.

Ex:-

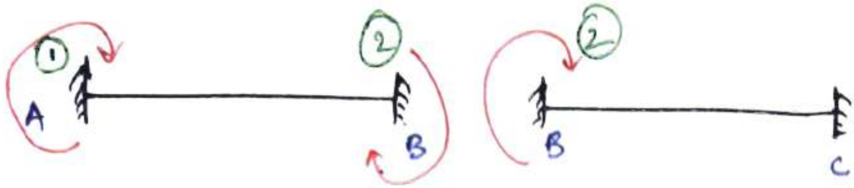


$D_k = 2\ (\theta_A, \theta_B)$ {Axial deformations are neglected}.

given co-ordinate number as shown in fig.

→

→ The restrained structure is as shown in fig.



Note :-

Since, θ_{BA} & θ_{BC} rotate in the same direction Co-ordinate ② for the span BA & BC must be in the same direction as shown in fig.

Step II :-

find force at all co-ordinates due to applied loads on the restrained structure.

$[P]_L$ {i.e., find fixed end Moment at all co-ordinates due to applied loads}.

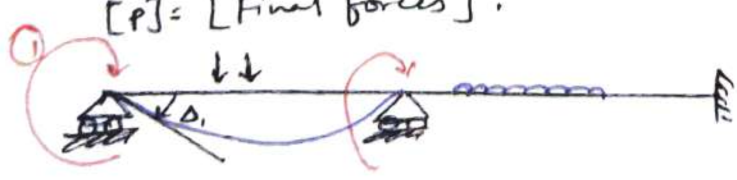
Step III :-

Find stiffness matrix [K].

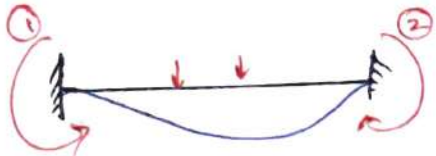
Step IV :-

Use equillibrium eq^n. to be find unknown displacement.

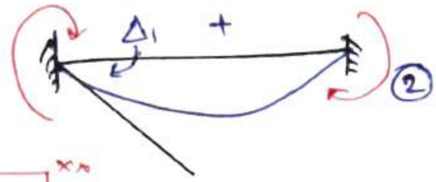
[P] = [Final forces].



To fix → $[P]_L$



To release → $[P]_\Delta$



$$[P] = [P]_L + [P]_\Delta$$ equilibrium eqⁿ.

$[P] = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ = final forces at all co-ordinates.

$[P]_L = \begin{bmatrix} P_{1L} \\ P_{2L} \end{bmatrix}$ = forces at all co-ordinates due to applied loads on the restrained structure.

$[P]_\Delta = \begin{bmatrix} P_{1\Delta} \\ P_{2\Delta} \end{bmatrix}$ = forces at all co-ordinate to produce displacement Δ_1, Δ_2 etc.

$$[P]_\Delta = [P] - [P]_L$$

we know that $[P]_\Delta = [K][\Delta]$

$$[\Delta] = [K]^{-1}[P]_\Delta$$

find displacement $[\Delta] = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}$ etc.

(v) Step:- After finding displacement use slope deflection eqⁿ. to find final end moments.

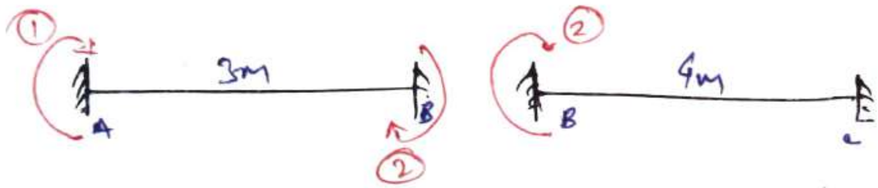
Q. Analyse the continous beam shown in fig. using stiffeness matrix MTD and Draw SFD and BMD.

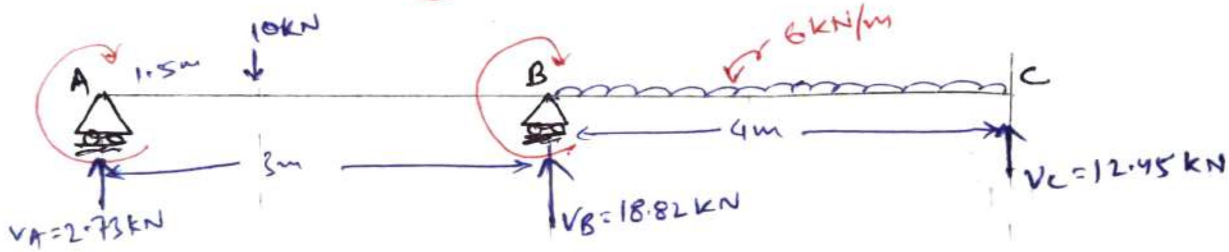
Sol^n:-

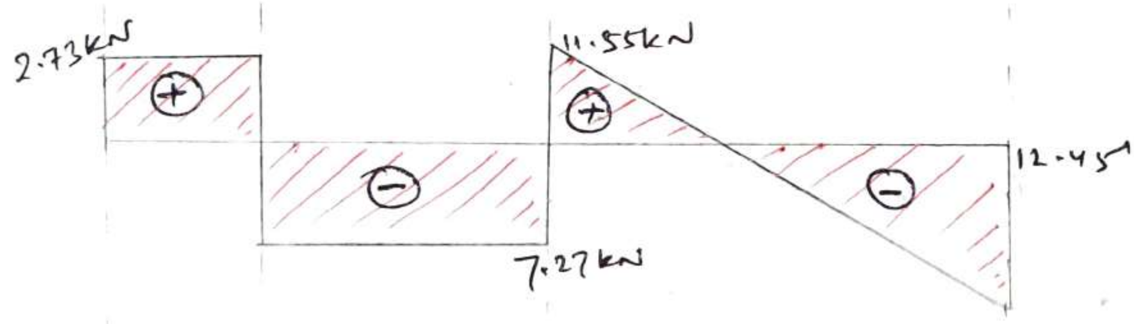
Step1:-

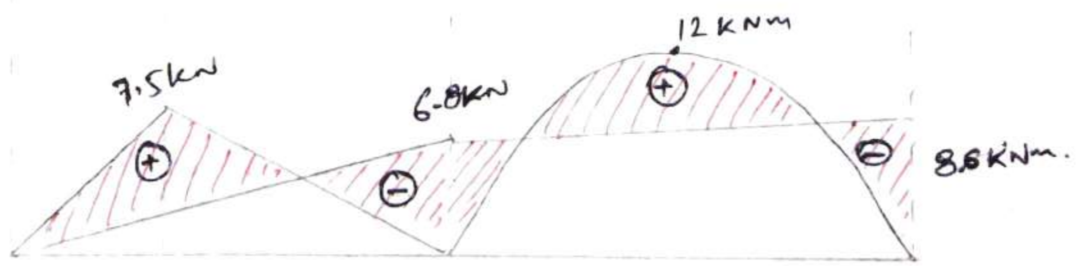
Find Dk, given co-ordinate number to displacements. fix all co-ordinate and get a restrained structure.

$D_K = 2\ (\theta_A, \theta_B)$. give coordinate number as shown in fig.









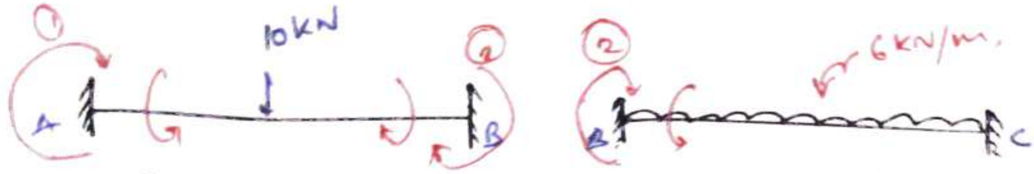
$\frac{Wl}{4} = \frac{10 \times 3}{4}$

$= 7.5\ KNm$

$\frac{wl^2}{8} = \frac{6 \times 4^2}{8} = 12\ KNm$

Step II :-

Find $[P]_L$ {Find fixed end Moments at all co-ordinate due to applied loads on the restrained structure}.



$M_{Fab} = P_{1L}$ $\quad$ $M_{Fab} = P_{21}$ $\quad$ $M_{Fbc} = P_{2L}$

Note:- If co-ordinate direction and FEM direction are same then take fem as +ve. If they are opposite to each other take f.e.m as −ve.

$\therefore P_{1L} = -\frac{wl}{8}$ {−ve bcz ① & M_{Fab} are opposite to each other}.

$= -\frac{10 \times 3}{8} = -3.75$ KNm.

$P_{2L} = \left[+\frac{wl}{8}\right]_{BA} + \left[\frac{wl^2}{12}\right]_{BC}$

$= \frac{+10 \times 3}{8} - \frac{6 \times 4^2}{8}$

$= -4.25$

So, $[P]_L = \begin{bmatrix} -3.75 \\ -4.25 \end{bmatrix}$

Step III :- Find Stiffeness Matrix [K] :-

I - Column :-

To get first column, apply unit displacement at co-ordinate 1 and find forces at all co-ordinat

$K_{11} = \frac{4EI}{l}$ (applied Moment)

$K_{21} = \frac{2EI}{l}$ {C.O.M}

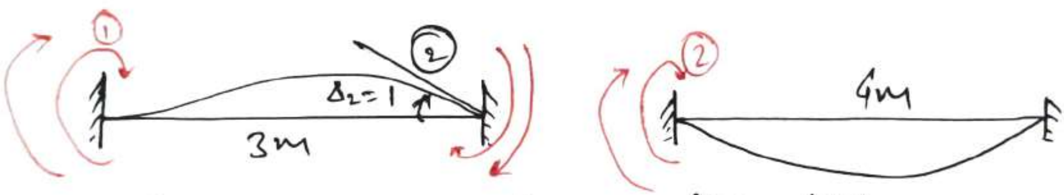
$[k] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$

$$K_{11} = \frac{4EI}{l} = \frac{4EI}{3}$$

$$K_{21} = \frac{2EI}{l} = \frac{2EI}{3}$$

(II) Column:-

To get 2nd column, apply unit displacement at $ co-ordinate (2) Find forces developed at all co-ordinate.



$K_{12} = \frac{2EI}{l}$ {C.O.M}

$K_{21} = \frac{4EI}{l}$ (applied moment)

$K_{22} = \frac{4EI}{l}$ {applied Moment}

$$K_{12} = \frac{2EI}{l} = \boxed{\frac{2EI}{3}}$$

$$K_{22} = \left[\frac{4EI}{l}\right]_{BA} + \left[\frac{4EI}{l}\right]_{BC} = \frac{4EI}{3} + \frac{4EI}{4} = \boxed{\frac{7EI}{3}}$$

$$\text{So, } [k] = \begin{bmatrix} \frac{4EI}{3} & \frac{2EI}{3} \\ \frac{2EI}{3} & \frac{7EI}{3} \end{bmatrix}$$

$$[K] = \frac{EI}{3}\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix} \Rightarrow \frac{3}{EI} \times \frac{1}{[28-4]}\begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix}$$

$$\Rightarrow \frac{1}{8EI}\begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix} = [K]^{-1}$$

Step :- Use equillibrium eq^n. to find unknown displacement :-

$$[P] = [P]_L + [P]_\Delta$$

$$[P] = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ { '0' bez, there are no external applied moment at co-ordinate ① and ② }.

$$[P]_L = \begin{bmatrix} -3.75 \\ -4.25 \end{bmatrix}$$

$$[P]_\Delta = [P] - [P_L] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -3.75 \\ -4.25 \end{bmatrix} = \begin{bmatrix} 3.75 \\ 4.25 \end{bmatrix}$$

$$= \frac{3}{EI(4\times7-2\times2)}\begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 3.75 \\ 4.25 \end{bmatrix}$$

$$[\Delta] = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} 2.22/EI \\ 1.19/EI \end{bmatrix} = \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix}$$

So, $\theta_A = \frac{2.22}{EI}$

$\theta_B = \frac{1.19}{EI}$

→

Step V :-

Use slope-deflection eq^n. to find final end Moment {Clockwise end moment +ve, Anticlockwise end Moment -ve}.

$$M_{AB} = M_{Fab} + \frac{2EI}{l}\left(2\theta_A + \theta_B - \frac{3\delta}{l}\right)$$

$$= \left[-\frac{10\times3}{8}\right] + \frac{2EI}{3}\left[2\times\frac{2.22}{EI} + \frac{1.19}{EI} - 0\right] = 0$$

$$M_{BA} = M_{Fba} + \frac{2EI}{l}\left(2\theta_B + \theta_A - \frac{3\delta}{l}\right)$$

$$= \left[+\frac{10\times3}{8}\right] + \frac{2EI}{3}\left[\frac{2\times1.19}{EI}\right] = 6.8\ kN\text{-}m.$$

$$M_{BC} = M_{Fbc} + \frac{2EI}{l}\left(2\theta_B + \theta_C - \frac{3\delta}{l}\right)$$

$$= \left[-\frac{6\times4^2}{12}\right] + \frac{2EI}{4}\left[2\times\frac{1.19}{EI}\right] = -6.8\ kN\text{-}m.$$

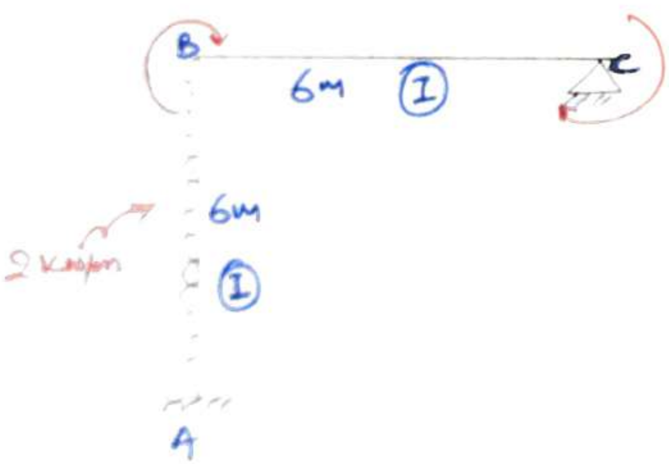
$$M_{CB} = M_{Fcb} + \frac{2EI}{l}\left(2\theta_C + \theta_B - \frac{3\delta}{l}\right)$$

$$= \left[+\frac{6\times4^2}{12}\right] + \frac{2EI}{4}\times\frac{1.19}{EI} = +8.6\ kN\text{-}m.$$

(VI). Step :- Support reaction S.F.D and B.M.D :-

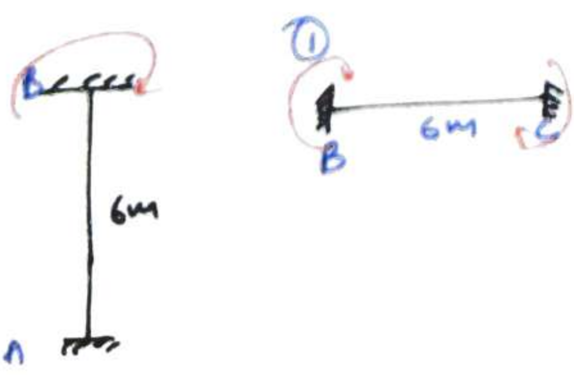
Due to loads	5 (↑)	5 (↑)	12 KN (↓)	12 KN (↑)
Due to moment-	2.27 (↓)	2.27 (↑)	0.45 (↓)	(0.45) (↑)
Final Reaction.	V_A = 2.73 KN	V_B = 18.82 KN (↑)		V_C = 12.45 KN (↑)

Q. Analyse the frame shown in fig. using stiffeness matrix MTD or displacement MTD?



Step 1 :-

D_k = 2 $\{\theta_B, \theta_C\}$ give co-ordinate number Restrained structure is.



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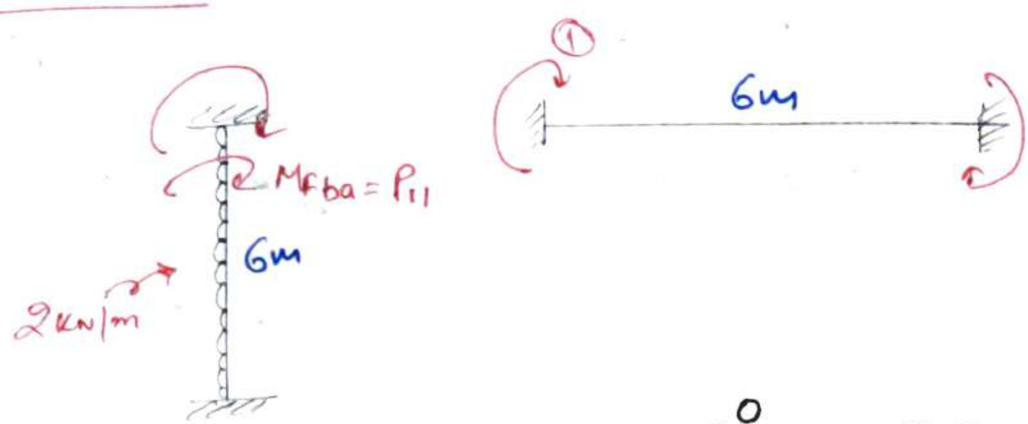
Step II :-

$[P]_L = \begin{bmatrix} P_{1L} \\ P_{2L} \end{bmatrix}$

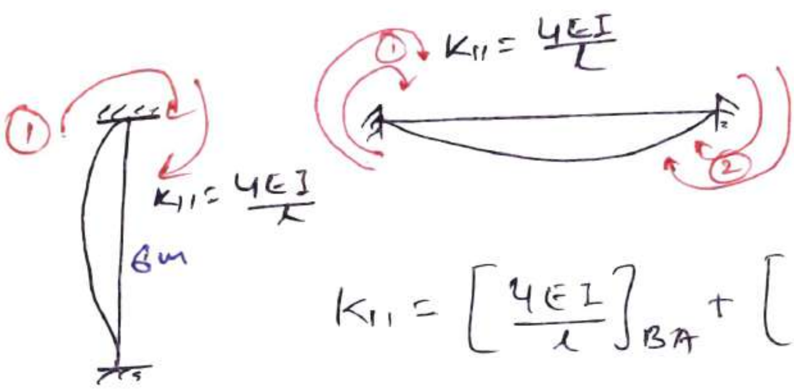


$$P_{11} = (P_{1L})_{BA} + \cancel{(P_{1L})_{BC}}^{\,0} \qquad \text{so } [P]_L = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$P_{1L} = +\frac{wl^2}{12} = +\frac{2 \times 6^2}{12} = 6 \text{ kNm}.$$

$P_{2L} = 0$ { No. load on span }.

Step III :- Stiffeness matrix [K] :-



$K_{21} = \frac{2EI}{l}$

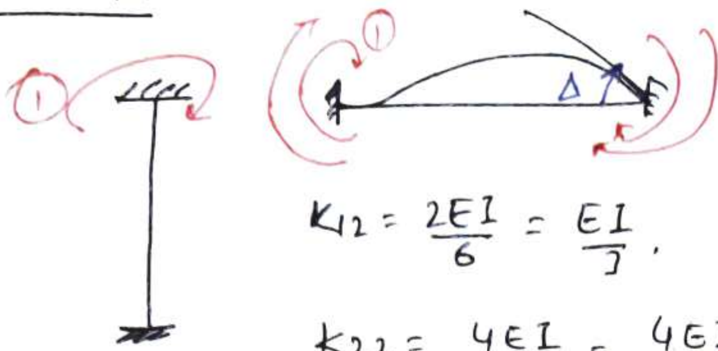
$$K_{11} = \left[\frac{4EI}{l}\right]_{BA} + \left[\frac{4EI}{l}\right]_{BC}$$

$$= \frac{4EI}{6} + \frac{4EI}{6}$$

$$= \boxed{\frac{4EI}{3}}$$

$$K_{21} = \frac{2EI}{6} = \boxed{\frac{EI}{3}}$$

II Column :-



$$K_{12} = \frac{2EI}{6} = \frac{EI}{3}.$$

$$K_{22} = \frac{4EI}{l} = \frac{4EI}{6} = \frac{2EI}{3}$$

So, $[K] = \begin{bmatrix} \frac{4EI}{3} & \frac{EI}{3} \\ \frac{EI}{3} & \frac{2EI}{3} \end{bmatrix}$

$$= \frac{EI}{3}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

Step iv Equi equations :-

$$[P] = [P]_L + [P]_\Delta$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} +6 \\ 0 \end{bmatrix} + \begin{bmatrix} P_{1\Delta} \\ P_{2\Delta} \end{bmatrix}$$

$$\Rightarrow [P]_\Delta = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$$

we know that, $[P]_\Delta = [K][\Delta]$

$$[\Delta] = [K]^{-1}[P]_\Delta = \frac{3}{EI(8-1)}\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} -6 \\ 0 \end{bmatrix}$$

$$[\Delta] = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{3}{7EI}\begin{bmatrix} -12 \\ 6 \end{bmatrix} = \begin{bmatrix} -5.13/EI \\ +2.57/EI \end{bmatrix}$$

$$\Delta_1 = \theta_B = -5.13/EI$$

$$\Delta_2 = \theta_C = \frac{2.57}{EI}$$

→

Step V :- Final end Moment :-

$$M_{AB} = M_{Fab} + \frac{2EI}{l}\left(2\theta_A^{0} + \theta_B - \frac{3\delta^{0}}{l}\right)$$

$$= \left[\frac{-2\times 6^2}{12}\right] + \frac{2EI}{6}\left[-\frac{5.13}{EI}\right]$$

$$= -7.71 \text{ KN.m}$$

-ve indicates anticlockwise.

$$M_{BA} = M_{Fba} + \frac{2EI}{l}\left(2\theta_B + \theta_A^{0} - \frac{3\delta^{0}}{l}\right)$$

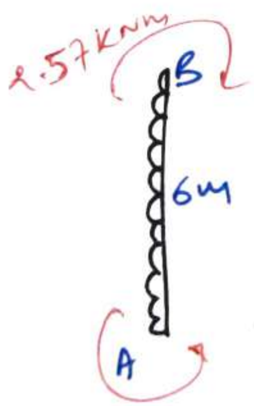
$$= \left[+\frac{2\times 6^2}{12}\right] + \frac{2EI}{6}\left[2\times\left(-\frac{5.13}{EI}\right)\right]$$

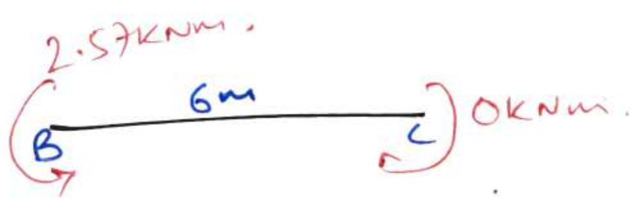
$$= 2.57 \text{ KNm}.$$

$$M_{BC} = -2.57 \text{ KNm}$$

$$M_{CB} = 0.$$

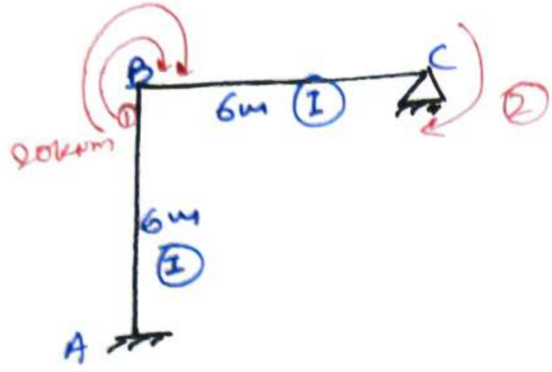
Step VI :- B.M.D :-





B.M.D is as usual.

Q.) Analyse the frame shown in fig. using stiffness Matrix MTD.



Step 1 :- Same as earlier.

Step 2 :- $[P]_L = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ { '0' bcz no applied loads on the span }.

Step 3 :- $[K]$ same as earlier.

Step 4 :- Equillibrium eqn. :-

$$[P] = [P]_L + [P]_\Delta$$

$[P] = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$ { +ve bcz 20 kN-m acting in the direction of ① }.

$$[P]_\Delta = [P] - [P]_L$$

$$= \begin{bmatrix} 20 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

We know, that-

$$[P]_\Delta = [K][\Delta]$$

$$[\Delta] = [K]^{-1}[P]_\Delta$$

$$= \frac{3}{EI(8-1)} \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

→

$$[\Delta] = \begin{bmatrix} 17.14/EI \\ -8.5/EI \end{bmatrix}$$

So, $\theta_B = \Delta_1 = \frac{17.14}{EI}$

$\theta_C = \Delta_2 = -\frac{8.57}{EI}$

Step V :- Final end Moments :-

$$M_{AB} = M_{fab}^{0} + \frac{2EI}{l}\left(2\theta_A^{0} + \theta_B - \frac{3\delta^{0}}{l}\right)$$

$$= \frac{2EI}{6}\left(\frac{17.14}{EI}\right) = 5.71 \text{ kNm}.$$

$$M_{BA} = M_{fba}^{0} + \frac{2EI}{l}\left(2\theta_B + \theta_A^{0} - \frac{3\delta^{0}}{l}\right)$$

$$= \frac{2EI}{6}\left[2 \times \frac{17.14}{EI}\right] = 11.42 \text{ kNm}.$$

$$M_{BC} = M_{fbc}^{0} + \frac{2EI}{l}\left(2\theta_B + \theta_C - \frac{3\delta^{0}}{l}\right)$$

$$= \frac{2EI}{6}\left[2 \times \frac{17.14}{EI} + \left(-\frac{8.57}{EI}\right)\right]$$

$$= +8.58 \text{ kN-m}.$$

$M_{CB} = 0$

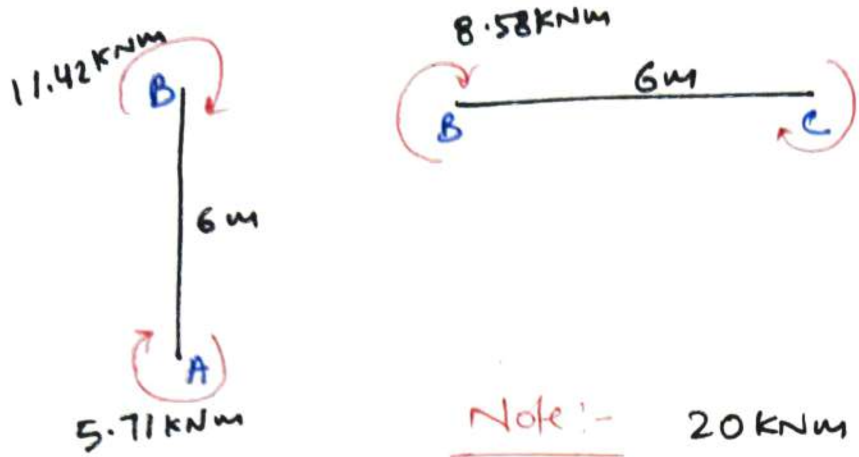
Check :-

$M_{BA} + M_{BC} = 20 \text{ kNm}.$

$11.42 + 8.58 = 20 \text{ kNm}.$

Results are correct.

Step VI :- B.M.D :-



Note :- 20KNM should not be kept on any f.b.d. It is shared by BA and BC as shown in fig.

Analysis of truss stiffness matrix MTD :-

Concept 1 :-

Stiffeness matrix for a truss joint is given by.

$$K_{11} = \Sigma \frac{AB}{L} \cos^2\theta$$

$$K_{22} = \Sigma \frac{AB}{L} \sin^2\theta$$

$$K_{12} = K_{21} = \Sigma \frac{AE}{l} \cos\theta \cdot \sin\theta$$

} 'θ' is always measured in anticlockwise direction x-axis.

(2) At a joint, stiffeness matrix is only a [2 × 2] matrix bez only 2 displacements are possible at a joint.

(3) Force in any member AB = F_{AB}

$$= -\frac{AE}{l}\left[(\Delta A_x - \Delta B_x)\cos\theta_{AB} + (\Delta A_y - \Delta B_y)\sin\theta_{AB}\right]$$

θ_{AB} = Angle of inclined of member AB measured in anticlockwise direction from x-axis.

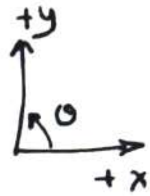
→

$\Delta Ax, \Delta Ay$ = x and y displacement of joint A

$\Delta Bx, \Delta By$ = x and y displacement of joint B.

Note :-

1) If Fab is +ve, it is a tensile force, and if FAB is -ve it is compressive force.

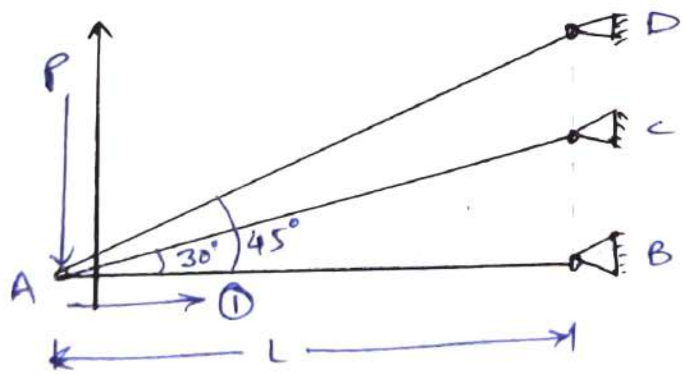


$\delta = \frac{Pl}{AE}$

$P = \frac{AE(\delta)}{l}$

2) co-ordinate ① and ② corresponding to displacement must be taken along +ve x-axis and +ve y-axis.

Q. Analyse the truss shown in fig. using stiffness matrix MTD.



$l_{AB} = L$

$l_{AC} = \frac{l}{\cos 30^\circ} = 1.15l$

$l_{AD} = \frac{l}{\cos 45^\circ} = 1.414l$

Sol^n:-

I Step:→ Find Dk give co-ordinate number

$D_K = 2\ (\Delta A_x, \Delta A_y)$

Co-ordinates must be along $+x$ axis and $+y$ axis as shown in fig.

II Step:- $[P]_L$ {fixed end Moment at all co-ordinates}.

Note:-

In trusses, there are no fixed end moments

so, $[P]_L = \begin{bmatrix} P_{1L} \\ P_{2L} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ always.

III Step:- Stiffeness Matrix [K] :-

$K_{11} = \Sigma \frac{AE}{l} \cos^2\theta$, $K_{12} = K_{21} = \Sigma \frac{AE}{l} \cos\theta \cdot \sin\theta$.

$K_{22} = \Sigma \frac{AE}{l} \sin^2\theta$.

θ ⇒ is always measured in anticlockwise from co-ordinate ①

Member	AE/l	θ	cosθ	sinθ	$\frac{AE}{l}\cos^2\theta$	$\frac{AE}{l}\sin^2\theta$	$\frac{AE}{l}\cos\theta\cdot\sin\theta$
AB	$\frac{AE}{l}$	0°	1	0	AE/l	0	0
AC	$\frac{AE}{1.15l}$	30°	0.866	0.5	$0.65\frac{AE}{l}$	$0.22\frac{AE}{l}$	$0.38\frac{AE}{l}$
AD	$\frac{AE}{1.414l}$	45°	0.707	0.707	$0.35\frac{AE}{l}$	$0.35\frac{AE}{l}$	$0.35\frac{AE}{l}$
				Sum	$2\frac{AE}{l}$ (K_{11})	$0.57\frac{AE}{l}$ (K_{22})	$0.73AE/l$ ($K_{12}=K_{21}$)

→

$$k = \begin{bmatrix} 2\frac{AE}{l} & 0.73\frac{AE}{l} \\ 0.73\frac{AE}{l} & 0.57\frac{AE}{l} \end{bmatrix}$$

$$= \frac{AE}{l}\begin{bmatrix} 2 & 0.73 \\ 0.73 & 0.57 \end{bmatrix}$$

IV step! -

Use equillibrium equation find unknown displacements.

$[P] = [P]_L + [P]_\Delta \Rightarrow$ Equi equations.

where, $[P] = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -P \end{bmatrix}$ find force at all co-ordinates.

-ve because it acts opposite to Co-ordinate ②

$$[P]_L = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[P]_\Delta = [P] - [P]_L = \begin{bmatrix} 0 \\ -P \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -P \end{bmatrix}$$

we know that,

$$[P]_\Delta = [K][\Delta]$$

$$\Rightarrow [\Delta] = [K]^{-1}[P]_\Delta$$

(V)

$$[\Delta] = \frac{l}{AE(2\times 0.57 - 0.73\times 0.73)} \begin{bmatrix} 0.57 & -0.73 \\ -0.73 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -P \end{bmatrix}$$

$$[\Delta] = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} \frac{1.19 Pl}{AE} \\ \frac{-3.3 Pl}{AE} \end{bmatrix}$$

So, $\Delta_1 = \Delta A_x = \frac{1.19 Pl}{AE}$

$\Delta_2 = \Delta A_y = \frac{-3.3 Pl}{AE}$ {-ve sign indicates ΔA_y acts opposite to Co-ordinate ② i.e; downwards}

(V) Step:- final forces in the members:-

$\Delta A_x = +\frac{1.19 Pl}{AE}$, $\Delta B_x = \Delta B_y = \Delta C_x = \Delta C_y = \Delta D_x = \Delta D_y = 0$ at joints B, C, D Because they are hinge supports.

$\Delta A_y = -\frac{3.3 Pl}{AE}$

$$F_{AB} = -\frac{AE}{l}\left[(\Delta A_x - \Delta B_x)\cos\theta_{AB} + (\Delta A_y - \Delta B_y)\sin\theta_{AB}\right]$$

$$= -\frac{AE}{l}\left[\left(\frac{1.19 Pl}{AE} - 0\right)\cos 0^\circ + \left(-\frac{3.3 Pl}{AE} - 0\right)\sin 0^\circ\right]$$

$F_{AB} = -1.19P$ {-ve sign indicates compressive forces}.

$$F_{AC} = -\frac{AE}{l}\left[(\Delta A_x - \Delta C_x)\cos\theta_{AC} + (\Delta A_y - \Delta C_y)\sin\theta_{AC}\right]$$

$$= -\frac{AE}{1.15 l}\left[\left(\frac{1.19 Pl}{AE} - 0\right)\cos 30^\circ + \left(-\frac{3.3 Pl}{AE} - 0\right)\sin 30^\circ\right]$$

$= -0.54P$ {+ve indicates tensile forces}.

→

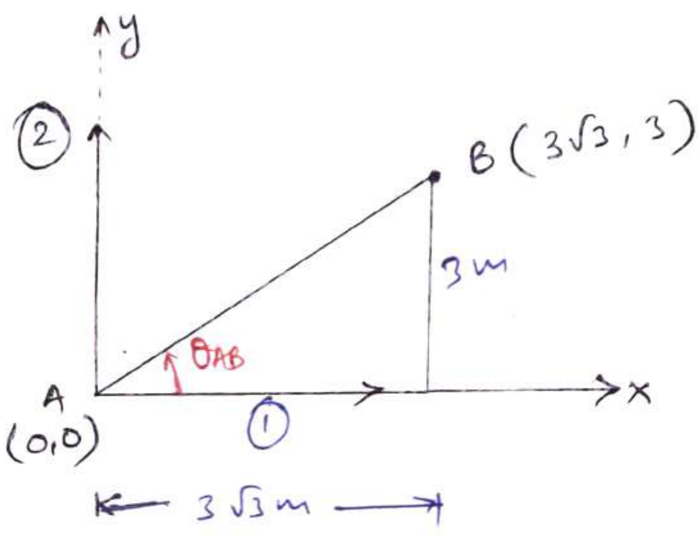
$$F_{AD} = -\frac{AE}{l}\left[(\Delta Ax - \Delta Bx)\cos\theta_{AB} + (\Delta Ay - \Delta Dy)\sin\theta_{AD}\right]$$

$$= \frac{-AE}{1.414l}\left[\left(\frac{1.19Pl}{AE} - 0\right)\cos 45^\circ + \left(-\frac{3.3Pl}{AE} - 0\right)\sin 40^\circ\right]$$

$$= 1.05P\,(T).$$

Q. (24) Page 91 workbook (2) Marks.

A member AB of a pin jointed truss lying in the x-y plane has co-ordinate A(0,0), B($3\sqrt{3}$, 3) m. Joints A & B are display and x and y components are (u_A, v_A, u_B, v_B) = (1, $\sqrt{3}$, 2, $2\sqrt{3}$) mm.
For this bar, $K = \frac{AE}{l} = 100$ kN/mm force induced in the bar by the end displacement is ?



$$\tan\theta_{AB} = \frac{3}{3\sqrt{3}} \Rightarrow \boxed{\theta_{AB} = 30^\circ}$$

$u_A = \Delta Ax = 1$ m	$u_B = \Delta Bx = 2$ mm
$v_A = \Delta Ay = \sqrt{3}$ mm	$v_B = \Delta By = 2\sqrt{3}$ mm.

$$f_{AB} = -\frac{AE}{l}\left[(\Delta Ax - \Delta Bx)\cos\theta_{AB} + (\Delta Ay - \Delta By)\sin\theta_{AB}\right]$$

$$= -100\left[(1-2)\cos 30^\circ + (\sqrt{3} - 2\sqrt{3})\sin 30^\circ\right]$$

$$f_{AB} = +173.1 \text{ KN}$$

(Difference in x- co-ordinates × cos θ) + Difference in y co-ordinates × sin θ

(*) Flexibility Matrix MTD of Analysis :-

Concepts:-

1. Flexibility (δ) :-

> It is the displacement produced due to unit force
It is the inverse of stiffeness.

2. Characteristics of flexibility matrix [δ] :-

(i) To get 1st column of flexibility matrix apply unit force at co-ordinates (1) and find displacement at all co-ordinates.

Similarly,

To get 2nd column of flexibility matrix apply unit force at co-ordinates (2) and find displacement at all co-ordinates.

(ii) Flexibility matrix is also a square matrix [Symmetric bcz of maxwell's reciprocal deflection theorem].

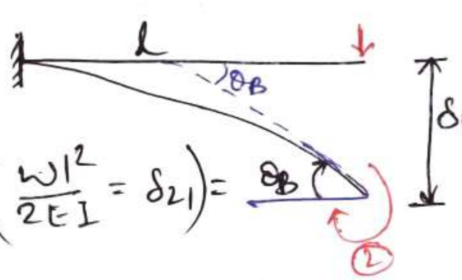
→

(IV) If structure is unstable large deformations will happen and flexibility matrix will not exists.

Q. [2-Marks].
{IES MASTER 2014} {15 marks}.

Sol^n:-

I column :- To get 1st column, apply unit force at ① & find displacement at all co-ordinates.



$\left(\frac{Wl^2}{2EI} = \delta_{21}\right) = \theta_B$

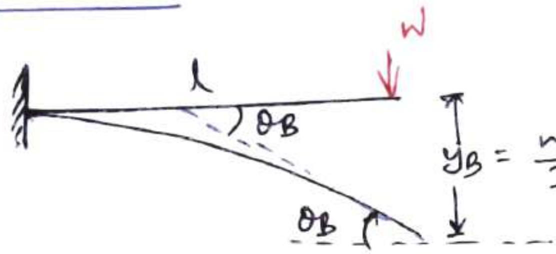
$\delta_{11} = \frac{Wl^3}{3EI} = \frac{l^3}{3EI}$

δ_{11} = Displacement at ① due to unit force at ① $= \frac{Wl^3}{3EI} = \frac{l^3}{3EI}$

δ_{21} = Displacement at ② due to unit force at ① $= \frac{Wl^2}{2EI} = \frac{l^2}{2EI}$

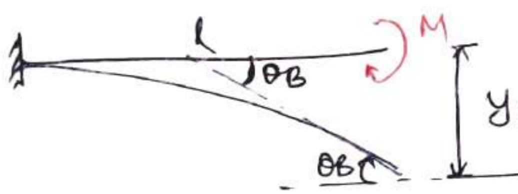
{ +ve because co-ordinate ② and θ_B are in same direction }.

From S.M :-



$$y_B = \frac{Wl^3}{3EI}$$

$$\theta_B = \frac{Wl^2}{2EI}$$

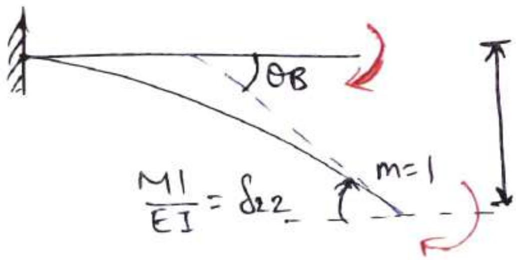


$$y_B = \frac{Ml^2}{2EI}$$

$$\theta_B = \frac{Ml}{EI}$$

II column :-

To get 2nd column apply unit force {i.e., unit moment} at ② and find displacements at all coordinates.



$$\frac{Ml}{EI} = \delta_{22}$$

$$\delta_{12} = \frac{Ml^2}{2EI} = \frac{l^2}{2EI} \quad (M = 1)$$

δ_{22} = Displacement at ② due to unit force at ②

$$= \frac{Ml}{EI} = \frac{l}{EI}$$

δ_{12} = Displacement at ① due to unit force at ②

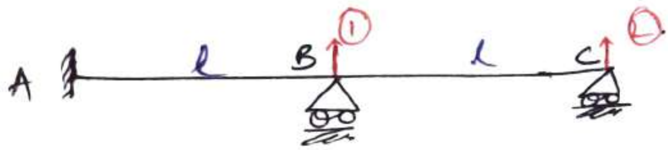
$$= \frac{Ml^2}{2EI} = \frac{l^2}{2EI}$$

→

So, $[\delta] = \begin{bmatrix} \frac{l^3}{3EI} & \frac{l^2}{2EI} \\ \frac{l^2}{2EI} & \frac{l}{EI} \end{bmatrix}$

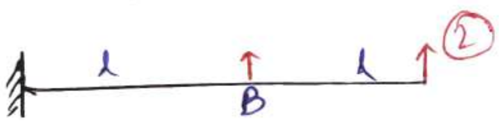
Q. [2 Marks].

what is the value of flexibility co-efficient F_{12} for the continous beam shown in fig.?



Sol^n:-

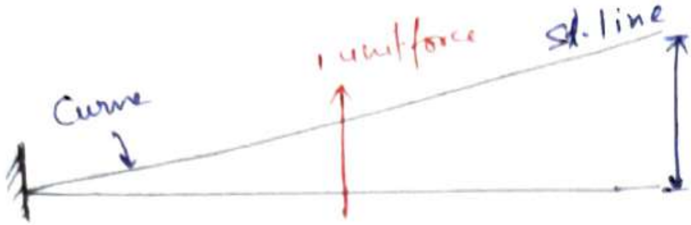
$f_{12} = \delta_{12}$ = Displacement at ① due to unit force at ② {after removing redundant reaction}.



$F_{12} = \delta_{12} = ?$

from Maxwell's reciprocal deflection theorem.

$$\boxed{\delta_{12} = \delta_{21}}$$



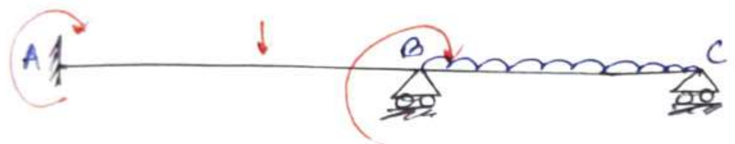
$$\delta_{21} = \frac{W l^3}{3EI} + \frac{W l^2}{2EI}(2l - l)$$

$$\delta_{12} = \delta_{21} = \frac{l^3}{3EI} + \frac{l^3}{2EI} = \frac{5l^3}{6EI}$$

⪢ Procedure for Flexibility Matrix MTD :-

Step 1 :- Find D_s choose redundent forces. give co-ordinate numbers to redundant. get released structure.

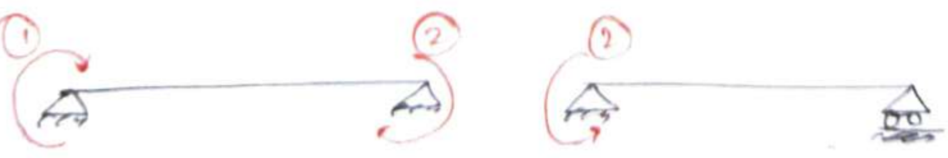
ex :-



$D_B = r - s$
$= 5 - 3 = 2$ { always choose moments as redundants, So that calculating deflections will be simple }.

Choose M_A and M_B as redundants. Give co-ordinate number to moments as shown in fig.

The release structure is



Note :- Since, M_{BA} and M_{BC} are opposite to each other at joint 'B'. co-ordinate ② for the span BA and BC must be opposite as shown in fig.

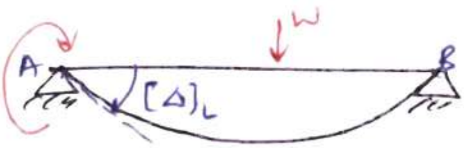
→

Step II :- Find $[\Delta]_L$. {find displacement at all co-ordinates due to applied loads on the released structure}

Step III :- Find Flexibility matrix $[\delta]$.

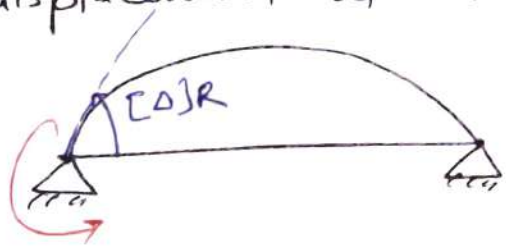
Step IV :- use compatibility condition to find redundant forces.

$[\Delta] = [\Delta]_L + [\Delta]_R$



$[\Delta]$ = final displacement at all components.

$[P]_R$



$[\Delta]_L$ = Displacement at all co-ordinates due to applied loads on the released structure.

$[\Delta]_R$ = Displacement at all co-ordinates due to redundant force $[P]_R$.

$[P]_R$ = Redundant force i.e, M_A, M_B etc.

$$[\Delta]_R = [\Delta] - [\Delta]_L$$

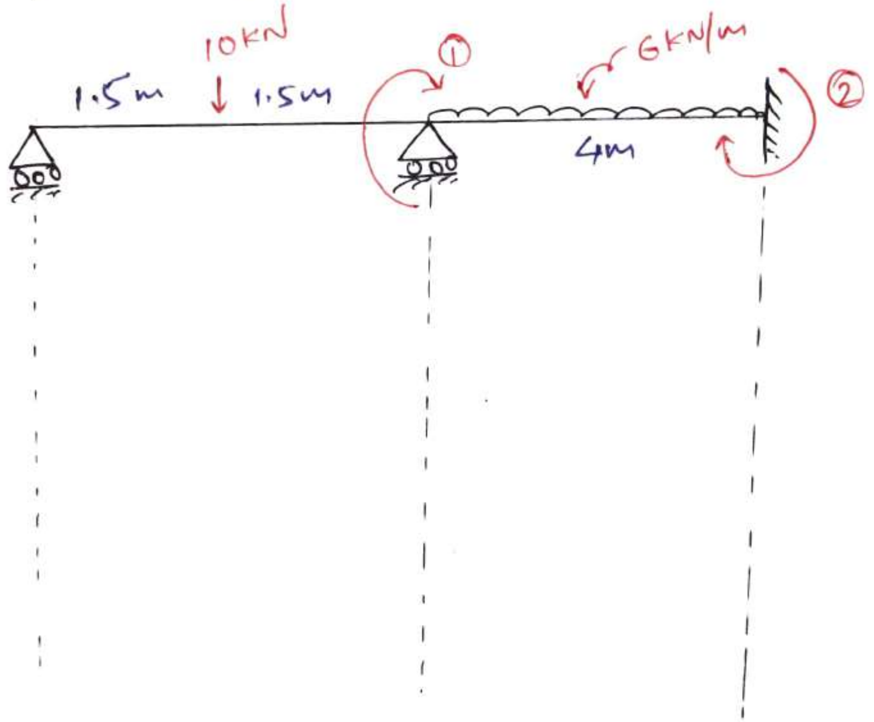
we know that $[\Delta]_R = [\delta][P]_R$

$$[P]_R = [\delta]^{-1}[\Delta]_R$$

Find redundant forces.

Q [Marks - 20].

using flexibility matrix MTD analyse the structure.



→

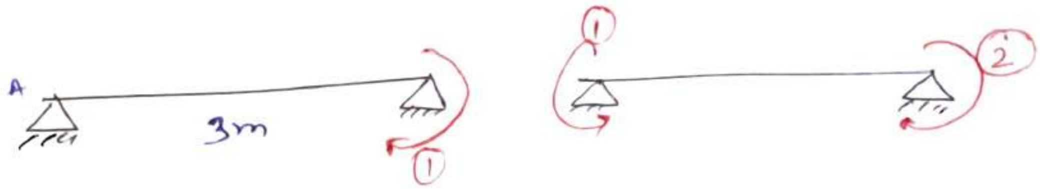
Solⁿ: Step I :-

- Find Ds. choose redundants give co-ordinate number. get released structure.

$D_S = r - S = 5 - 3 = 2$ (choose M_B and M_C as redundants).

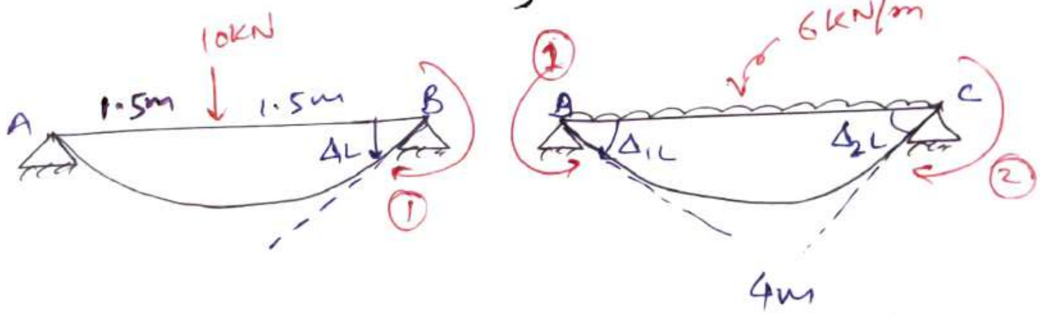
give co-ordinate number as shown in fig.

Released structure is.



II step :-

Find $[\Delta]_L$ { Displacement at all co-ordinate due to applied load on the release structure }.



Note :- If co-ordinate direction and displacement direction are same, then take displacement as +ve.

$$\Delta_{1L} = \theta_B = \left[\frac{-wl^2}{16EI}\right]_{BA} + \left[\frac{-wl^3}{24EI}\right]_{BC}$$

$$\Delta_{1L} = -\frac{10\times3^2}{16EI} - \frac{6\times4^3}{24EI} = \frac{-21.625}{EI}$$

$$\Delta_{2L} = \frac{-10\times 3^2}{16EI} - \frac{6\times 4^3}{24EI} = \frac{-21.625}{EI}$$

$$\Delta_{2L} = \frac{-wl^3}{24EI}$$ {-ve bcz ② and Δ_{2L} are opposite to each other}.

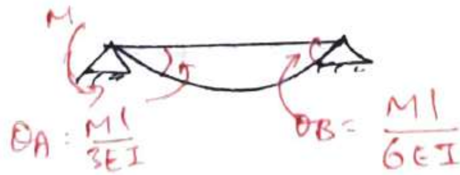
$$= \frac{-6\times 4^3}{24EI} = \frac{-16}{EI}$$

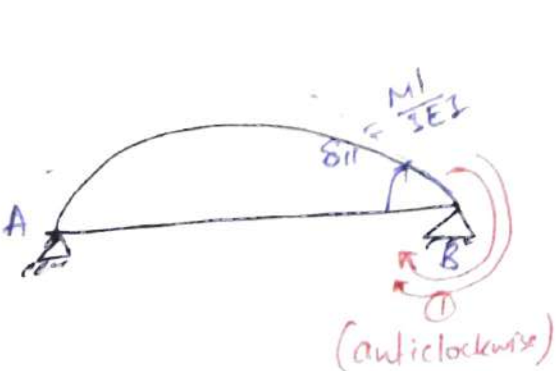
So, $[\Delta]_L = \begin{bmatrix} -\frac{21.625}{EI} \\ -\frac{16}{EI} \end{bmatrix}$

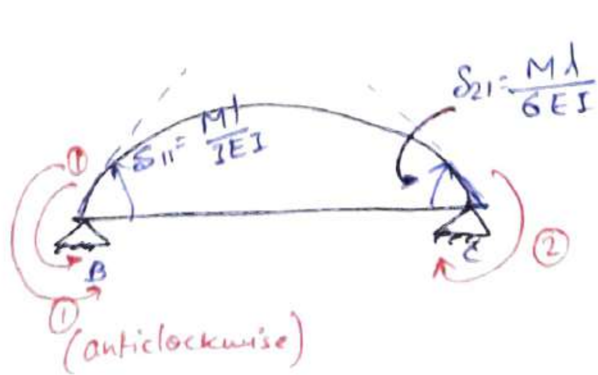
Step III :- Find Flexibility Matrix [δ] :-

I - Column :-

To get 1st column apply unit force at ① and find displacement at all coordinates.







→

δ_{11} = Displacement at ① due to unit force at ①

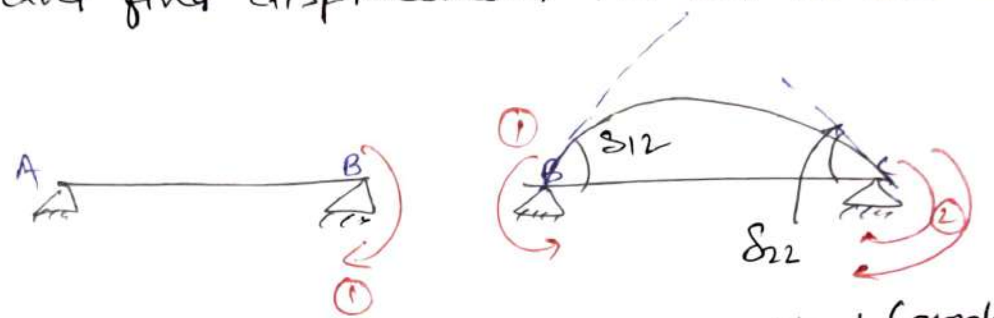
$$= \left[\frac{\overset{=1}{M} l}{3EI}\right] + \left[\frac{\overset{=1}{M} l}{3EI}\right] = \frac{3}{3EI} + \frac{4}{3EI} = \boxed{\frac{7}{3EI}}$$

$\delta_{21} = +\frac{Ml}{6EI}$ { +ve because displacement at ② and co-ordinate ② are in the same direction }.

$$= \frac{4}{6EI} = \frac{2}{3EI}$$

II column :-

To get 2nd column, apply unit force at ② and find displacement at all co-ordinates.



M = 1 (applied moment)

δ_{22} = Displacement at ② due to unit force at ②

$$= \frac{Ml}{3EI} = \frac{4}{3EI}$$

δ_{12} = Displacement at ① due to unit force at ②

$$= \frac{Ml}{6EI} = \frac{4}{6EI} = \frac{2}{3EI}.$$

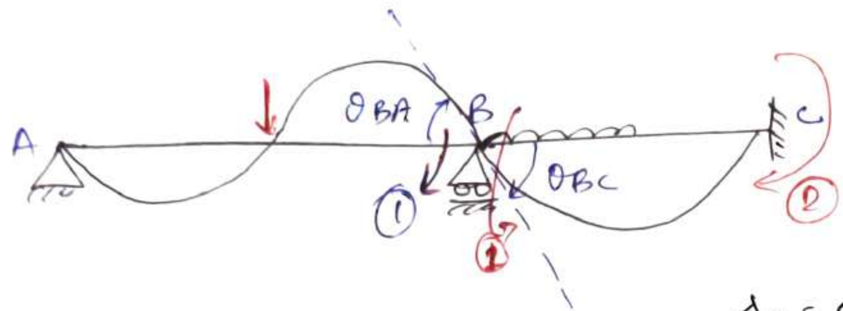
IV step :-

use compatibility condition to find unknown redundants :-

$[\Delta] = [\Delta]_L + [\Delta]_R \Rightarrow$ Compatibility condition.

$[\Delta] = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: final displacement at all co-ordinates.

explanation :-



$\Delta_2 = 0$ (bcz of fixed support)

$\boxed{\theta_C = 0}$

$\Delta_1 = [+\theta_{BA}]_{BA} + [-\theta_{BC}]_{BC} = 0$

+ve bcz θBA and ① are in same direction.

-ve bcz θBC & ① are in opposite direction.

$\Delta_2 = 0$ bcz of fixed supports.

So, $[\Delta]_R = [\Delta] - [\Delta]_L = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -21.625/EI \\ -16/EI \end{bmatrix}$

$= \begin{bmatrix} 21.625/EI \\ 16/EI \end{bmatrix}$

we know that

$[\Delta]_R = [\delta][P]_R$

$[P]_R = [\delta]^{-1}[\Delta]_R$

$\delta = \begin{bmatrix} \frac{7}{3EI} & \frac{2}{3EI} \\ \frac{2}{3EI} & \frac{4}{3EI} \end{bmatrix}$

$= \frac{1}{3EI}\begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$ →

$$= 3EI \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 21.625/EI \\ 16/EI \end{bmatrix}$$

$$= \frac{3EI}{(7\times4-2\times2)} \begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} \frac{21.625}{EI} \\ 16/EI \end{bmatrix}$$

$$= \frac{3EI}{8} \begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} \frac{21.625}{EI} \\ 16/EI \end{bmatrix}$$

$$[P]_B = \begin{bmatrix} P_{1R} \\ P_{2R} \end{bmatrix} = \begin{bmatrix} M_B \\ M_C \end{bmatrix} = \begin{bmatrix} 6.8 \\ 8.6 \end{bmatrix}$$

$M_B = +6.8$ KN-m
$M_C = +8.6$ KNm.

} +ve sign indicates moments acts in the direction of Co-ordinates.

Results are s as in stiffeness MTD. Draw S.F.D and B.M.D as earlier.

$M_{AB} = 0$
$M_{BA} = +6.8$ kN-m
$M_{BC} = -6.8$ kN-m
$M_{CB} = +8.6$ kN-m

} Ans.

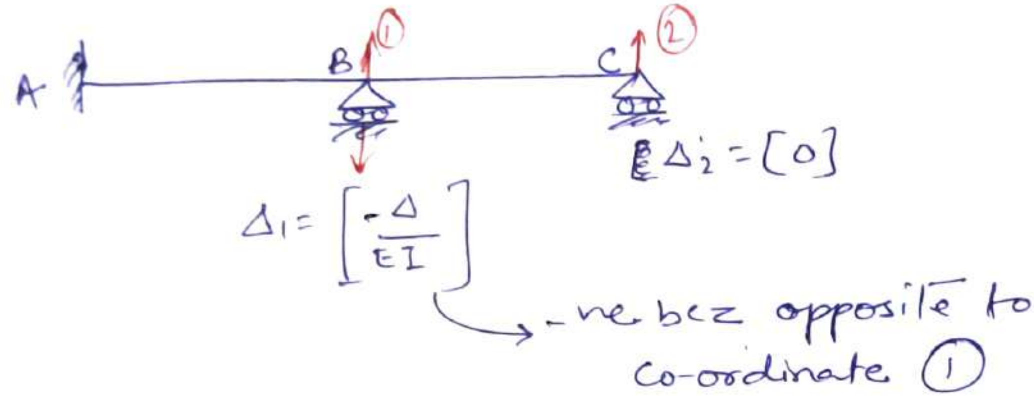
Q. [Marks-2].

Flexibility matrix for the beam shown in fig as.

$$[\delta] = \frac{1}{3EI}\begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}$$

It supports 'B' settles by $\frac{\Delta}{EI}$ units.

What is the reaction at B?



Sol^n:

III step:-

Compatibility condition.

$$[\Delta] = [\Delta]_L + [\Delta]_R$$

$$\begin{bmatrix} -\frac{\Delta}{EI} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + [\Delta]_R$$

↳ [0 bcz there are no loads on span].

$$[\Delta]_R = \begin{bmatrix} -\Delta/EI \\ 0 \end{bmatrix}$$

We know that ⇒ $[\Delta]_R = [\delta]\cdot[P]_R$

$$[P]_R = [\delta]^{-1}[\Delta]_R.$$

→

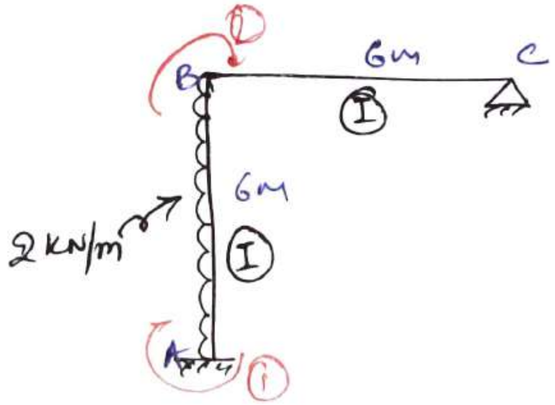
$$= \frac{3EI}{8-4}\begin{bmatrix} 8 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -4/EI \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} P_{1R} \\ P_{2R} \end{bmatrix} = \begin{bmatrix} V_3 \\ V_C \end{bmatrix} = \frac{3EI}{4}\begin{bmatrix} -8\Delta/EI \\ +\frac{2\Delta}{EI} \end{bmatrix} = \begin{bmatrix} -6\Delta \\ +1.5\Delta \end{bmatrix}$$

So, $\boxed{V_B = -6\Delta}$, $\boxed{V_C = 1.5\Delta}$

Q. [Marks – 20]

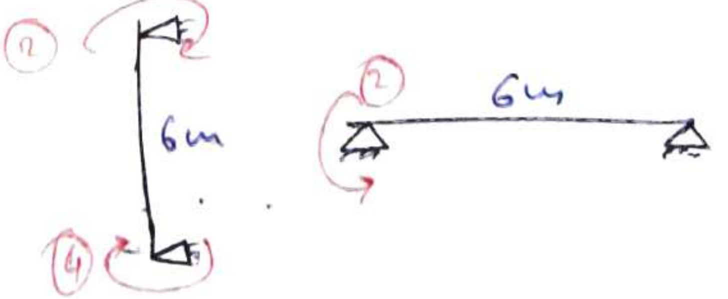
Analyse the force shown in fig using Compatibility MTD, Draw S.F.D & BMD.



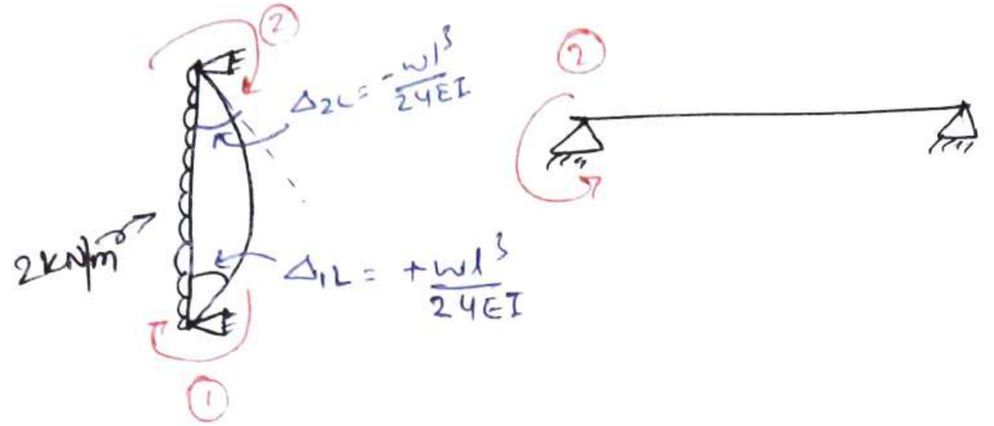
Sol^n :-

Step 1 :- $D_s = r - s = 5 - 3 = 2$

Choose M_A and M_B as redundants.



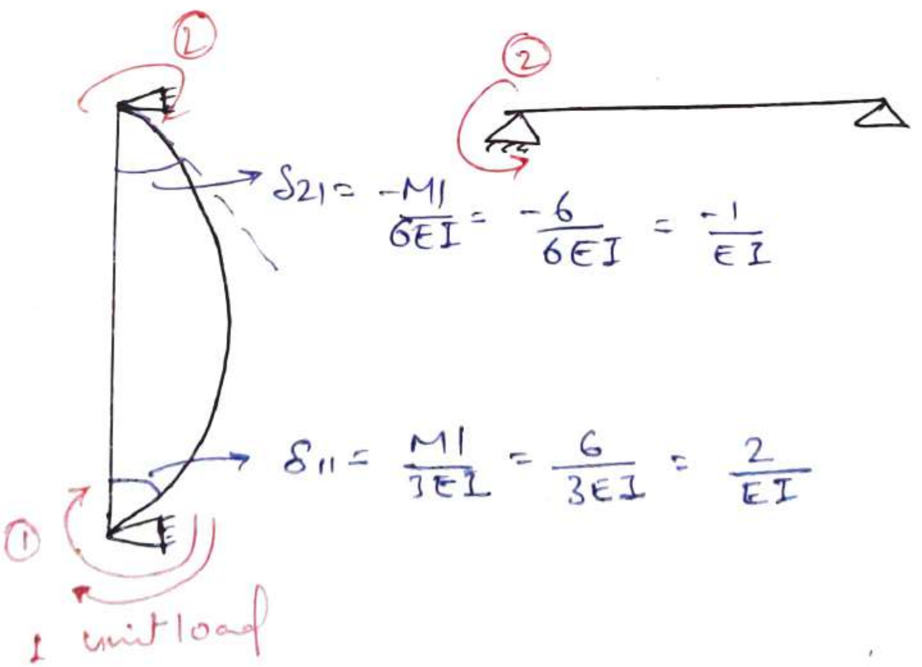
II step :- $[\Delta]_L$:⇒



$$\Delta_{1L} = \frac{2 \times 6^3}{24EI} = \frac{18}{EI}$$

$$\Delta_{2L} = -\frac{18}{EI}$$

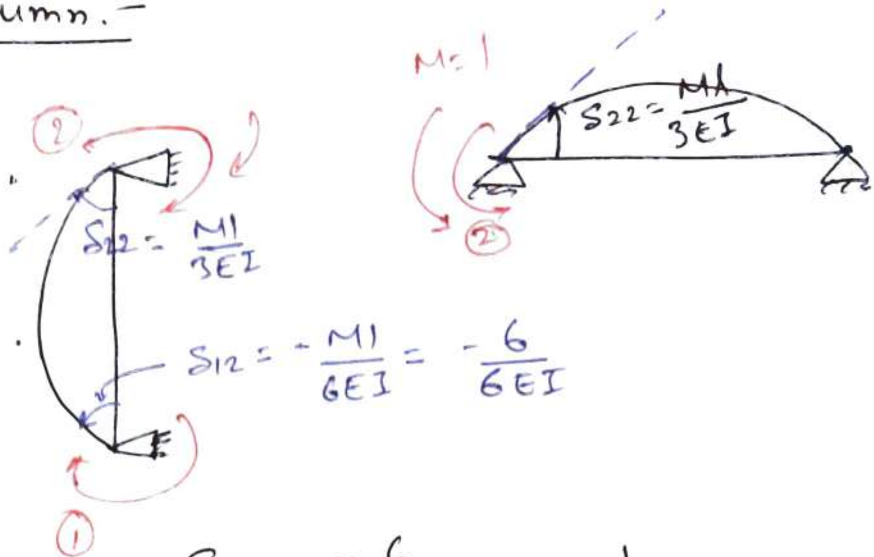
III step :- $[\delta]$



$$\delta_{11} = \frac{2}{3EI} \Rightarrow \delta_{21} = \frac{-1}{EI}$$

→

II column:-



$$\delta_{12} = -\frac{6}{6EI} = -\frac{1}{EI}$$

$$\delta_{22} = \left[\frac{l}{3EI}\right]_{BA} + \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

IV Step:- use compatibility conditions to find unknown displacement.

$$[\Delta] = [\Delta]_L + [\Delta]_R$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} +18/EI \\ -18/EI \end{bmatrix} + [\Delta]_R \Rightarrow [\Delta]_R = \begin{bmatrix} -18/EI \\ +18/EI \end{bmatrix}$$

We know that,

$$[\Delta]_R = [\delta] \cdot [P]_R$$

$$[P]_R = [\delta]^{-1} [\Delta]_R$$

→

$$= \frac{EI}{(4\times2-1\times1)} \begin{bmatrix} 4 & +1 \\ +1 & 2 \end{bmatrix} \begin{bmatrix} -18/EI \\ +18/EI \end{bmatrix}$$

$$[P]_R = \begin{bmatrix} P_{1R} \\ P_{2R} \end{bmatrix} = \begin{bmatrix} M_A \\ M_B \end{bmatrix} = \begin{bmatrix} -7.71 \\ +2.57 \end{bmatrix}$$

$M_A = -7.71$ KN.m { -ve sign implies opposite ①

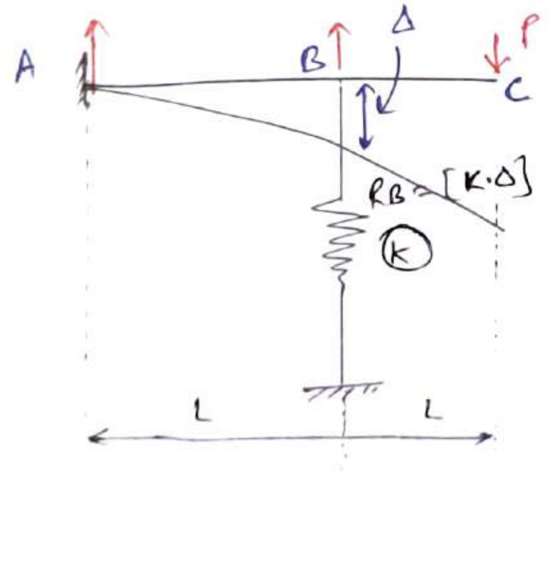
$M_B = 2.57$ KN.m { the sign implies is the direction ② }

Results are same as in stiffeness MTD.

Q. [Marks - 20]. (16/99)

work book :-

using force MTD Analyse the structure shown in fig.

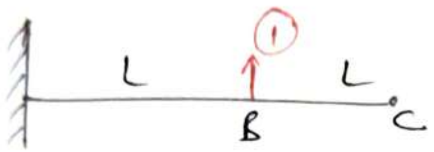


Step I :-

$$Ds = r - s$$
$$= 4 - 3 = 1$$

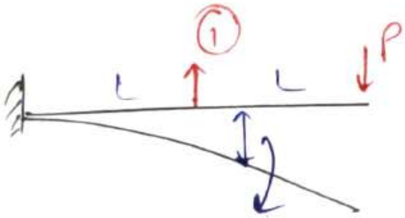
Choose, R_B as redundent. give co-ordinate ① as shown in fig.

Released structure is :-



Step II :-

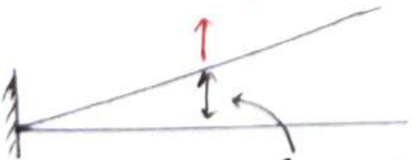
$[\Delta]_L$



$$\Delta_{1L} = -\frac{5PL^3}{6EI}$$

$$[\Delta]_L = \left[-\frac{5PL^3}{6EI} \right]$$

Step III :- $[\delta]$



$$\delta_{11} = \frac{wl^3}{3EI} = \frac{l^3}{3EI}$$

$$[\delta] = \left[\frac{l^3}{3EI} \right]$$

Step IV :

Compatibility condition :-

$$[\Delta] = [\Delta]_L + [\Delta]_R$$

$$[-\Delta] = \left[-\frac{5Pl^3}{6EI}\right] + [\Delta]_R$$

-ve bcz opposite to ①

$$[\Delta]_R = \left[-\Delta + \frac{5Pl^3}{6EI}\right]$$

$$[\Delta]_R = [\delta] \cdot [P]_R$$

$$[P]_R = [\delta]^{-1} [\Delta]_R$$

$$= \left[\frac{3EI}{l^3}\right] \left[-\Delta + \frac{5Pl^3}{6EI}\right]$$

$$[P_R] = [R_B] = [R_B] = -\frac{3EI\Delta}{l^3} + \frac{5P}{2}$$

$$\Rightarrow R_B = K\Delta = -\frac{3EI\Delta}{l^3} + \frac{5P}{2}$$

$$\Delta\left[K + \frac{3EI}{l^3}\right] = \frac{5P}{2}$$

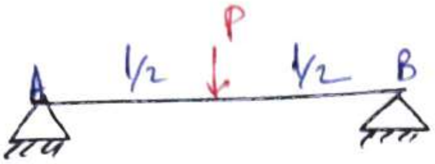
$$\Delta\left[\frac{Kl^3 + 3EI}{l^3}\right] = \frac{5P}{2}$$

$$\Delta = \frac{5Pl^3}{2(KL^3 + 3EI)}$$

$$R_B = K\Delta = \frac{5Pl^3 \textcircled{\Delta} \rightarrow K}{2(KL^3 + 3EI)}$$

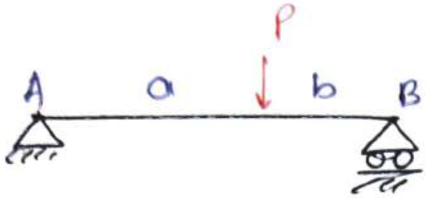
→

Slope at ends: -



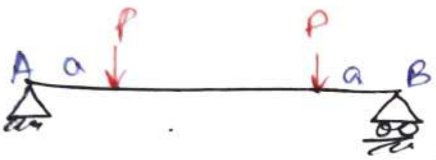
$\theta_A = \theta_B = \dfrac{Pl^2}{16EI}$

*



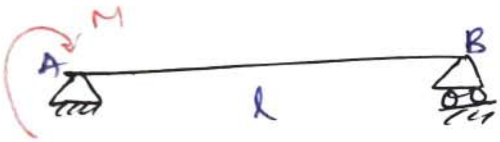
$\theta_A = \dfrac{Pab(l+b)}{6lEI}$, $\theta_B = \dfrac{Pab(l+a)}{6lEI}$

*

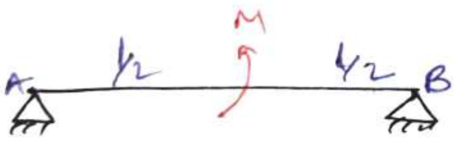


$\theta_A = \theta_B = \dfrac{Pa(l-a)}{2EI}$

→ $M_{fab} = M_{fba} = \dfrac{Pa(l-a)}{l}$



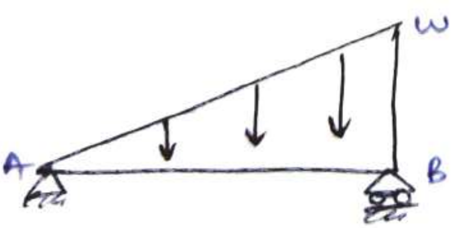
$\theta_A = \dfrac{Ml}{3EI}$, $\theta_B = \dfrac{Ml}{6EI}$



$\theta_A = \theta_B = \dfrac{Ml}{24EI}$ | $M_{fab} = M_{fba} = M/4$

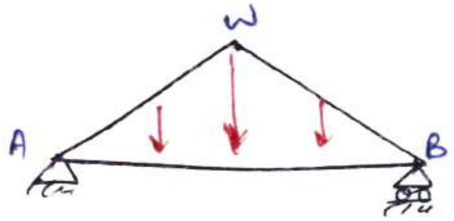
$\theta_A = \theta_B = \dfrac{Ml}{2EI}$

*



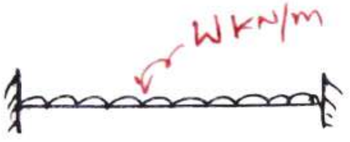
$M_{fab} = -\dfrac{wl^2}{30}$, $M_{fba} = +\dfrac{wl^2}{20}$

$\theta_A = \dfrac{7wl^3}{360EI}$, $\theta_B = \dfrac{wl^3}{45EI}$



$$\theta_A = \theta_B = \frac{5Wl^3}{192EI}$$

$$M_{fab} = M_{fba} = \frac{5}{96}Wl^2 \qquad M_{max} = \frac{Wl^2}{12}$$



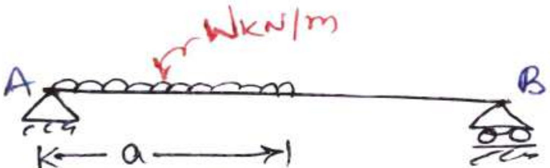
$$\theta_A = \theta_B = \frac{wl^3}{24EI}$$

*

$$M_{fab} = \frac{Mb(3a-l)}{l^2} \quad ; \quad M_{fba} = \frac{Ma(3b-l)}{l^2}$$

$$\theta_{AB} = \frac{M}{6lEI}\left(6al - 3a^2 - 2l^2\right)$$

$$\theta_{BA} = \frac{M}{6lEI}\left(3a^2 - l^2\right)$$

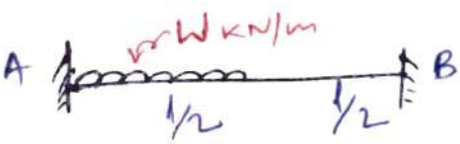


$$\theta_A = \frac{wa^2}{24lEI}(2l-a)^2$$

$$\theta_B = \frac{wa^2}{24lEI}(2l^2 - a^2)$$

$$M_{fab} = \frac{wa^2}{12l^2}\left[6l^2 - 8la + 3a^2\right]$$

$$M_{fba} = \frac{wa^3}{12l^3}(4l - 3a)$$



$$M_{fab} = -\frac{11}{192}wl^2 \quad ; \quad M_{fba} = \frac{5}{192}wl^2$$