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Discrete mathematics

Module I.

Mathematical logic: Statements and notations, connectives, normal form, theory of inference for the statement calculus. The Predicate calculus. Theory of predicate calculus.

Module II

Set theory: concept of set theory. representation of discrete structure. Relation, ordering, functions

Algebraic structure: Algebraic system, semi groups and monoids, groups, lattices as partially ordered sets. Boolean Algebra

Module III

Elementary combination: Basic of counting, combinations and Permutations, Enumeration of combinations and Permutations, Enumerating combinations and permuations with repeatitions with constrained repeatitions, Binomial coefficients, the Binomial and multinomial theorem. The principle of inclusion - Exclusion.

Module IV
Recurrence relation: Generating functions of sequence, calculating coefficients of generating functions. Recurrence relation, solving recurrence relation by substitution and generating functions. The method of characteristic roots, solutions of inhomogeneous R.R.

Module V
Isomorphism and subgraphs. Trees and their properties spanning trees. Directed trees, Binary trees, Planar graphs, Euler's formula; Multigraphs and Euler circuits, Hamilton graph chromatic numbers. The four colour Theorem.

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Module ?

Mathematical logic

Fundamentals of logic (statement and notation)

1) $2 < 5$ True statement
2) $5 < 2$ False statement
3) Where are you? (Not st. intrrogating)

Basic connectives and truth tables:

1. Negation (Not)

Symbol = $\sim, \neg, \neg$

Truth table :

P	$\sim P$
T	F
F	T

2. Conjunction (AND)

$P \quad Q \quad P \wedge Q$

Symbol = $\wedge$

Truth table

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

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Disjunction (OR)

P Q $P \vee Q$

Symbol = $\vee$

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional operation (if ... then)

$p \rightarrow q$ if p then q

Antecedent (hypothesis)

Consequent (conclusion)

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p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$p \rightarrow q \equiv \sim p \vee q$

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forms of conditional operator :-

conditional statement $p \to q$

i) Statement : $p \to q$ (if p then q)
ii) Converse : $q \to p$ (if q then p)
iii) Inverse : $\neg p \to \neg q$ (if not p then not q)
iv) contrapositive : $\neg q \to \neg p$ (if not q then not p

Q. P = "The home team wins whenever it is raining"

Ans. if it is raining then home team wins
Converse: if home team wins then it is raining
Inverse : if it is not raining then home team does not wins
Contrapositive:- if home team does not win then it is not raining

Biconditional operator (if and Only if)

$p \leftrightarrow q$ is true when p and q both have the same truth value.

$$p \leftrightarrow q \cong p \to q \wedge q \to p$$

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
T	T	T	T	T
T	f	f	T	F
F	T	T	F	F
F	F	T	T	T

Q. p : You can take the flight
q : You buy a ticket, the $p \leftrightarrow q$ is?

Ans You can take the flight if and only if you buy a ticket

compound proposition

Precedence of logical operator :

Precedence	operator
1	$\neg$
2	$\wedge$
3	$\vee$ and $\oplus$
4	$\rightarrow$
5	$\leftrightarrow$

Q: Construct the truth table of this compound proposition.
$(p \vee \neg q) \rightarrow (p \wedge q)$

Ans

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow p \wedge q$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

$p \rightarrow q \wedge \neg r \leftrightarrow r \oplus q$

p	q	r	$\neg r$	$q \wedge \neg r$	$p \rightarrow q \wedge \neg r$	$r \oplus q$	final
T	T	T	F	F	F	F	T
T	T	F	T	T	T	T	T
T	F	T	F	F	F	T	F
T	F	F	T	F	F	F	T
F	T	T	F	F	T	F	F
F	T	F	T	T	T	T	T
F	F	T	F	F	T	T	T
F	F	F	T	F	T	F	F

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Relations:- Relations are derived from cartesian product.

$A = \{1,2\}$, $B = \{a,b,c\}$

$A \times B = \{(1,a), (1,b), (1,c)\ (2,a) (2,b), (2,c)\}$

Let A and B be the two non-empty sets, then a relation R from A to B is a subset of $A \times B$.

$$R \subseteq A \times B$$

Ex: $A = \{1,2\}$, $B = \{1,2,3\}$

$R = \{$ $m = 2$, $n = 3 \Rightarrow 2^{2 \cdot 3} = 2^6$

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$R_{max} = A \times B$, $R_{min} = \{\phi\}$ Empty set.

Total no. of Relations $= 2^{m \cdot n}$

Where m = No. of elements of set A

n = No. of elements of set B.

Binary Relation:- A Binary relation 'R' from A to B is a set of ordered pairs where first element is from A and the second element is from set B..

$A = \{a, b\}$, $B = \{1, 2, 3\}$.

ARB ; $R = \{(a,1), (a,2), (b,3)\}$

Types of Relations

i) **Reflexive relation** :- Let R be a relation in set A. Then, R is called Reflexive relation

if $(a,a) \in R$ for all $a \in A$.

Let $A = \{1, 2, 3, 4\}$

$a \to 1, 2, 3, 4$

(a,a) $\{(1,1), (3,3), (2,2), (4,4)\}$ diagonal

R_1: $\{(1,1), (2,4), (3,3), (4,1), (4,4)\}$

$\{(1,1) (2,2) (2,3) (3,3) (4,4)\}$

No — $(2,2) \notin R_1$ — Not a reflexive relation

Yes — reflexive relation

$R_3 = \phi$] No

$R_4 = A \times A$] Yes

ii) **Irreflexive Relation** :- Relation R on a set A is said to be irreflexive if $a \in A$, $(a,a) \notin R$. It is also called anti Reflexive relation

$A : \{1,2,3,4\}$ $\{(1,1), (2,2), (3,3), (4,4)\}$

R_1

$\{(1,2), (2,1), (3,3)$
$(4,4)\}$

Not an irreflexive rel^n

R_2

$\{(1,2), (1,3), (2,4)\}$

Yes Irreflexive rel^n.

$R_1 = \phi \}$ Yes

$R_2 = A \times A \}$ No

Symmetric Relation :- Let R be a rel^n in set A. Then, R is said to be symm rel^n.

if $(a,b) \in R \Rightarrow (b,a) \in R$ $\{\phi$ = is sym rel$^n\}$

$xRy = yRx$.

$A = \{1,2,3,4\}$, $R_1 = \{(1,2), (2,1), (3,4), (4,3)\}$

$R_2 = \{(1,3), (3,1), (3,4), (4,4)\}$ $\times^n$

$R_3 = \{(1,1), (2,2), (3,3), (4,4)\}$

symm rel^n.

Anti-Symmetric Relation :- A relation R on a set A is said to be anti-symm if $(a,b) \in R$ and $(b,a) \in R$. then

$$a = b$$

$(1,2)(2,1)$ ✗ but $(1,1)$

$$a = b$$

$A = \{1,2,3,4\}$, $R = \{(1,2), (3,3), (4,3)\}$

$3 = 3 \rightarrow a = b$

Imp Ques :- Can a rel^n be symm or anti symm both?

Ans. Yes, $\because A = \{a,b,c\}$

$R_1 = \{(a,a)(b,b)(c,c)\}$

Symmetric → Anti Symm

$\because (a,b) \in R \Rightarrow (b,a) \in R$ $\quad$ $\because (a = b)$ for $\forall (a,b) \in R$

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Asymmetric Relation:- A relation R on a set A is said to be Asymmetric relation if

$(a,b) \in R \Rightarrow (b,a) \notin R$

$A = \{1,2,3,4\}$

$R_1 = \{(1,2),(2,3),(3,1),(4,3)\}$

Yes

$R_2 = \{(1,2),(2,1),(2,3),(4,3)\}$

NO

$R = \{(1,1),(2,2),(3,3),(4,4)\}$

Not Asymmetric

$R = A \times A$] NO

$R = \phi$] empty set]

Transitive Relations :- A relation R on a set A is said to be transitive if $(a,b) \in R$, $(b,c) \in R$, then $(a,c) \in R$ $a-b-c$.

i) $\{(1,2),(2,1),(1,1),(2,2)\}$ Yes $1-2-1$, $2-1-2$

ii) ~~$\{(1,1),(2,2),(3,3)\}$ Yes~~

iii) $\{(1,2)\}$ Yes

iv) $\{(1,2),(1,3)\}$ Yes $1-2$, $1-3$

v) $\{(1,2),(3,2)\}$ Yes $1-2$, $3-2$

vi) ~~$\{(1,2),(2,3),(1,3),(1,1)\}$~~ Yes $1-2-3$, $2-3$ ×

Inverse Relation :- Let R be a relation from set A to set B, then the Inverse rel^n R^{-1} from set B to the set A is defined by

$$\{(b,a) : (a,b) \in R\}$$

Q. Let $A = \{1,2,3\}$ and $B = \{a,b\}$
$R = \{(1,a),(1,b),(3,a),(2,b)\}$ be a rel^n from A to B.

$$R^{-1} = \{(a,1),(b,1),(a,3),(b,2)\}$$

Identity Relation :- A rel^n R in a set A is said to be identity rel^n (I_A),

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if $i_A = \{(x,x) : x \in A\}$

$A = \{2,4,6\}$

$i_A = \{(2,2),(4,4),(6,6)\}$

<u>Complement of a relation</u> :- R, R^c

$\bar{R} = \{(a,b) \mid (a,b) \in A \times B \text{ and } (a,b) \notin R\}$

$\bar{R} = (A \times B) - R$

$A = \{a,b\}$ and $B = \{1,2,3\}$, if

$R = \{(a,2),(b,1),(b,3)\}$. find $\bar{R}$

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$A \times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$

$\bar{R} = A \times B - R$

$\bar{R} = \{(a,1),(a,3),(b,2)\}$

<u>Equivalence Relation</u> :- (RST)

A relⁿ 'R' is equivalent relation in <u>set</u> 'A' if and only if

(1) R is reflexive, for all $a \in A$, $(a,a) \in R$

(ii) R is symmetric, $(a,b) \in R$ then $(b,a) \in R$

(iii) R is transitive, $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$

Q. $A = \{a, b, c\}$

$R_1 = \{\ \}$

$R_2 = \{(a,a), (b,b), (c,c)\}$ → R → S → T ✓

$R_3 = \{(a,a), (b,b), (c,c), (b,a)\}$ R✓ S✗ ✗

$R_4 = \{(a,a), (a,c), (b,a), (c,a)\}$ R✗ ✗

$R_5 = \{(a,a), (b,b), (c,c), (a,b), (a,c), (b,a), (c,a)\}$ R S T ✓

$R_6 = A \times A$ R S T ✓

Partial ORDER:- A relⁿ R on a set 'A' is said to be partial order if (RAT)

(i) R is reflexive, $(a,a) \in R, \forall a \in A$

(ii) R is anti-symmetric $\forall (a,b) \in A$ and $(a,b) \in R \wedge (b,a) \in R$ then $a = b$

(iii) R is transitive, $\forall (a,b,c) \in A$ and $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.

Q. $A = \{1,2,3\}$, $R = \{(1,1), (2,2), (3,3)\}$

$R_1 = \{\ \}$ ✗

$R_2 = \{(1,1), (2,2), (3,3)\}$ R A T ✓

$R_3 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$ A✗ ✗

$R_4 = \{(1,1), (2,2), (3,3), (1,3), (2,3)\}$ R A T ✓

$R_5 = \{(1,1), (1,2), (2,3), (1,3)\}$ ✗

$R_6 = A \times A$

$(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)$ - -

→ Not anti-symmetric

∴ Not Partial order.

Hasse diagram :- Representation of POSET :

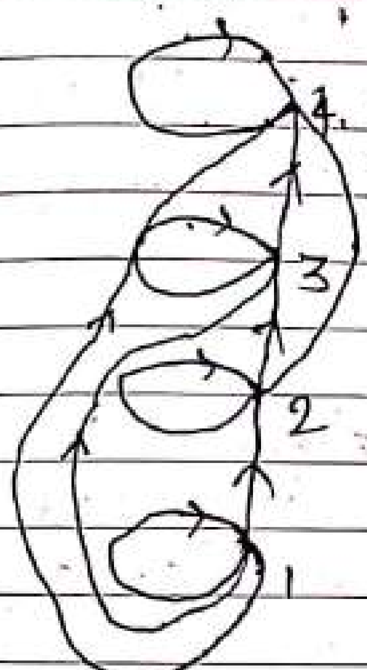
(i) Create vertex for element [upwards]

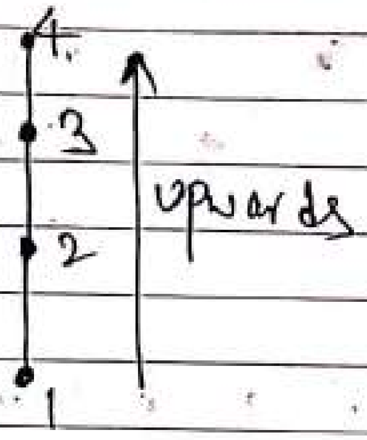
(ii) if aRb then draw edge from a to b

(iii) Remove self loop and transitive edges.

Imp

Ex :- Construct the Hasse Diagram for ({1, 2, 3, 4}, ≤)





Raw Hasse diagram . : Final Hasse diagram

Imp

Q. Draw the Hasse diagram representing the partial ordering {(a, b) | a divides b} on {1, 2, 3, 4, 6, 8, 12}

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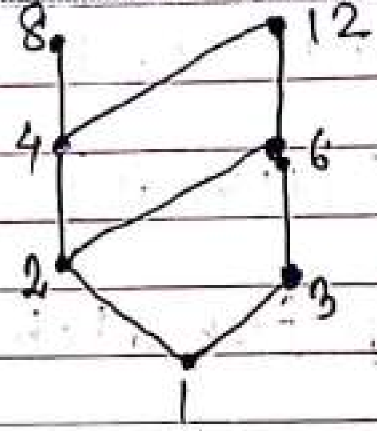
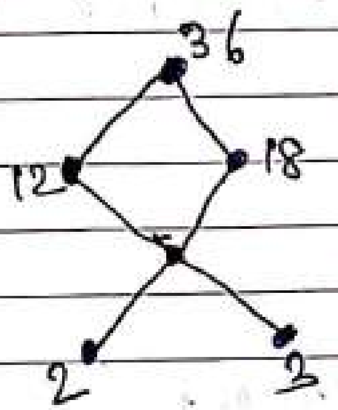


↑ upwards

(aRb)

Ex1

$A = \{2,3,6,12,18,36\}$ {a divides b}.



(2,6), (2,12), (2,18), (2,36), (3,6), (3,12), (3,18), (3,36), (6,12), (6,18), (6,36), (12,36), (18,36)

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Maximal and Minimal elements :

An element of poset is called maximal if it is not less than any element of the POSET.

a is maximal in the POSET (S, ≤) if there is no b∈S such that a<b

Imp [Maximal = Top elements of Hasse Diagram]

It is an element that is not related to any other element

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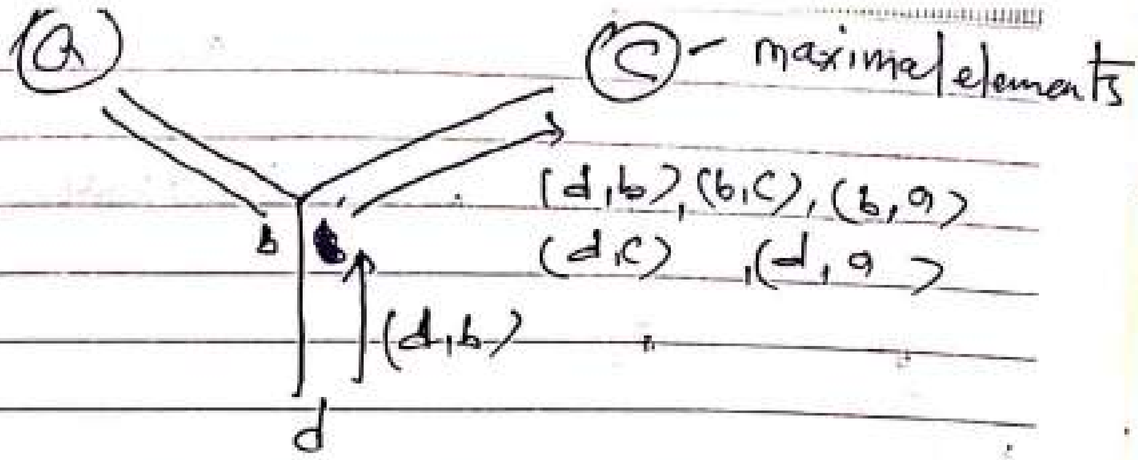
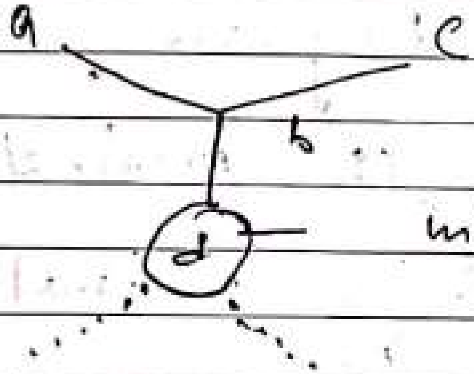


An element of a POSET is called as minimal if it is not greater than any element of the POSET.

a is minimal if there is no element $b \in S$, such that $b < a$

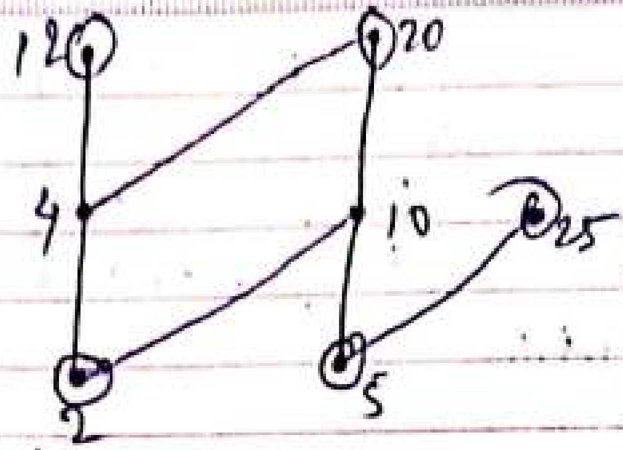
Imp.

[Minimal = Bottom element of the Hasse diagram]



minimal elements

Q. Which elements of the POSET $\{2, 4, 5, 10, 12, 20, 25\}$ are maximal and which are minimal.



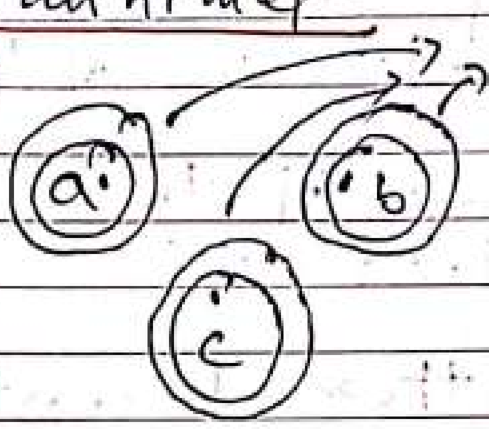
Maximal = 12, 20, 25

Minimal = 2, 5

Imp

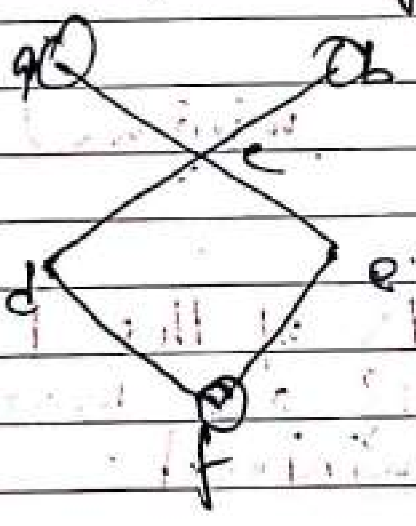
Q. Can a element be both maximal and minimal

Yes



(a, b, c) all are both maximal and minimal elements

Q. Find maximal and minimal element in following diagram

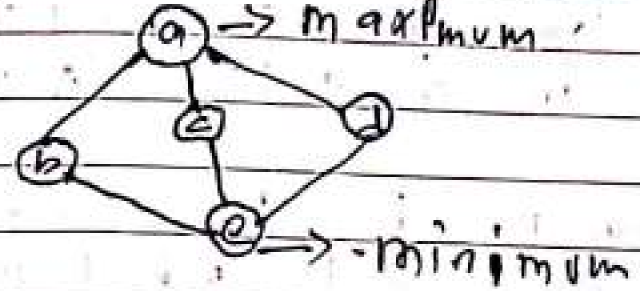


maximal = a, b

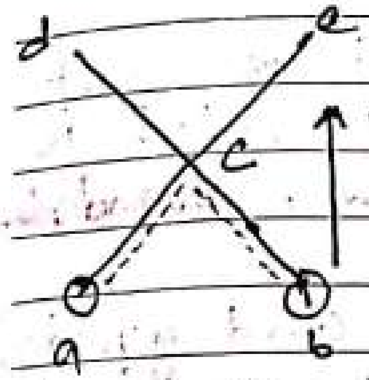
minimal = f

Maximum Element:- If it is maximal and every element is related to it.

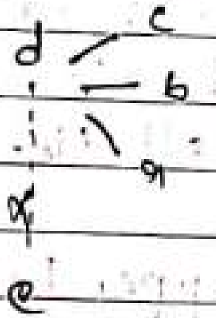
Minimum Element:- If it is minimal and it is related to every element in POSET.

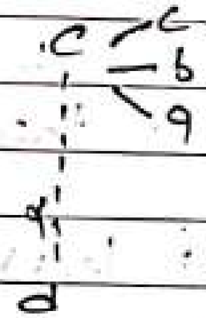


Q. Find maximum and minimum element.

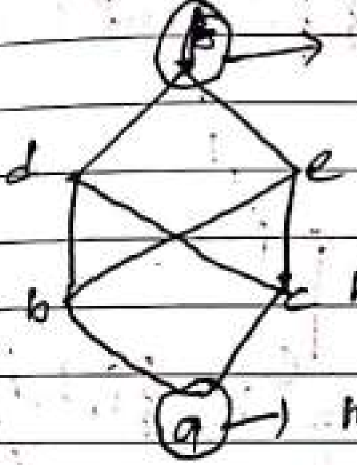


maximal = (d,e) no maximum





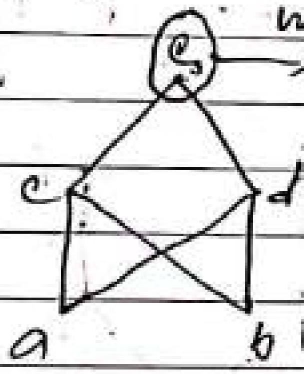
element
No max^m
element



maximal = F
maximum

minimal = a

minimum



maximal = e
maximum

minimal = a,b

No minimal element

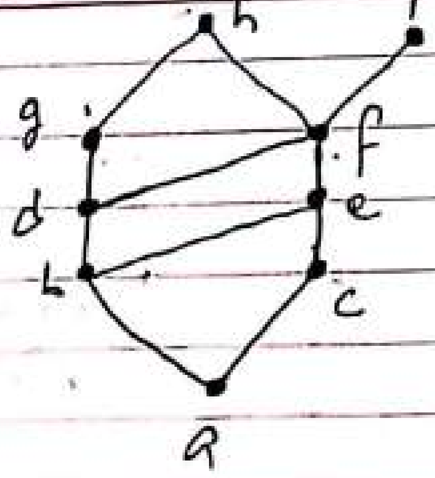
Upper-Bound :- These are the elements that are greater than or equal to all the elements in a subset 'A' of Poset 'S'.

Lower Bound :- These are the elements that are lower than or equal to all the elements in a subset 'A' of Poset 'S'

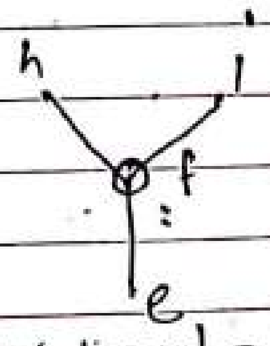
Least upper Bound :- It is one of the upper bound elements which is less than all the other upper-bound elements.

Greatest Lower-bound :- It is one of the lower bound elements which is greater than all the lower bound elements

Q. Find least upper bound and greatest lower bound of subset {b,c} and {e,f}.



(e,f)

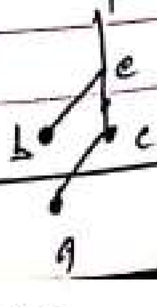


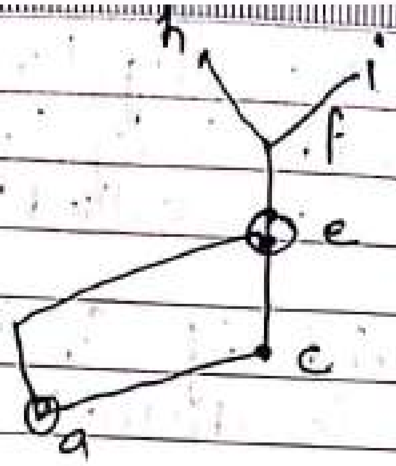
Upper bound = {f, h, i}

LUB = f

LB = e, c, b, a

GLB = e





(b, c) → upper bound = (e, f, h, i)

LUB = e

Lower bound = a

GLB = a

Algebraic System :- An algebraic system is a system that contains a non-empty set A and one or more operations on the set A.

If the operations are defined as $f_1, f_2, \ldots$ then the Algebraic system

$$(A, f_1, f_2, \ldots)$$

In addition if we consider some relⁿ $(R_1, R_2, \ldots)$ defined the set A

$(A, f_1, f_2, \ldots, R_1, R_2 \ldots)$ then this system form an Algebraic structure.

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Groupoid or Binary Algebra:-
A non empty set G equipped with one binary $(\cdot)$ is called groupoid.

G is a groupoid if G is closed for $*$. It is denoted by $(G, *)$.

Ex. $(N,+), (Z,-), (Q,\times)$.

It is called Groupoid or Quasi Group.

$N = \{1, 2, 3, 4, \ldots\}$

$3, 4 \in N$

$3+4 \in N$ | $3-4 \notin N$

$3 \times 4 \in N$ | $3 \div 4 \notin N$

$Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

-2 and $3 \in Z$ | $-2, -3 \notin Z$

$-2 + 3 \in Z$

$-2 \times 3 \in Z$

Semi group :-

An algebraic structure $(G, *)$ is called a semi group if the binary operation $*$ satisfies associative property.

$(G, *)$. $(a * b) * c = a * (b * c) \ \forall a \in G$

Ex. The algebraic structure $(N, +)$, $(Z, +)$, $(Z, \times)$, $(Q, \times)$ are semi groups but the structure $(Z, -)$ is not so because subtraction $(-)$ is not associative.

Ex. The structure $(P(S), \cup)$ and $(P(S), \cap)$ where $P(S)$ is power set of a set S are semi groups as both the operations union $(\cup)$ and intersection $\cap$ are associative.

Monoid :-

A semi group is called monoid if there exists an identity element 'e' in G such that:

$$e * a = a * e = a \ \forall a \in G$$

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Ex. The semi group $(N, \times)$ is a monoid because 1 is the identity for multiplication. But the semigroup $(N, +)$ is not, because 0 is the identity for addition is not in N.

Ex. The semi groups $(P(S), \cup)$ and $(P(S), \cap)$ are monoid because ϕ and S are the identities for union $(\cup)$ and intersection $(\cap)$ in P(S)

Group:-

An algebraic structure of set G and a binary operation * defined in G. The operation $(G, *)$ is called group if it satisfies the following cond'n

1) closure: $a \in G, b \in G \Rightarrow a * b \in G, \forall a, b \in G$

2) Associativity: The operation * in G is associative

$$(a * b) * c = a * (b * c) \quad \forall a, b, c \in G$$

3) Existence of identity:

There exist an identity element e in G such that

$$e * a = a * e = a \quad \forall a \in G$$

4) Existence of inverse :-

Each element of G is invertible, for every $a \in G$. there exist a^{-1} in G such that

$$a * a^{-1} = a^{-1} * a = e \text{ (identity)}$$

$\therefore (G, *)$ is a monoid in which each of its element is

Abelian group (commutative group) :-

A group $(G, *)$ is said to be abelian or commutative if $*$ is commutative also A group $(G, *)$ is an abelian group if.

i) Commutativity :-

$$a * b = b * a \quad \forall\, a, b \in G$$

ii) Associativity :- Semi group

iii) Existence of identity - Monoid

iv) Existance of inverse :- Group

Finite and infinite group:-

A group $(G, *)$ is said to be finite if its underlying set G is a finite set and a group which is not finite is called an infinite group.

Order of a group:-

The number of elements in a finite group is called the order of the group.

It is denoted by $O(G)$

If $(G, *)$ is an infinite group then it is said to be infinite order.

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Generating Function :-

Let $a_0, a_1, a_2, \ldots \infty$ be a series of real numbers denoted as $\{a_n\}$. then a series in power of x such that

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \infty$$

$$g(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$$ is called

generating function series $\{a_n\}$

Uses

1) To solve for counting problems

2) To solve recurrence relations.

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Q. Find the generating function for following series.

$$1, -1, 1, -1, \ldots \infty$$

Soln. We have

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \infty \quad \text{---(1)}$$

$a_0 = 1$, $a_1 = -1$, $a_2 = 1 \ldots$

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then $g(x) = 1 - x + x^2 - x^3 + \cdots$

$$= \frac{1}{1+x}$$

$$
\begin{array}{l}
1+x \,)\, 1 \,(\, 1 - x + x^2 \\
\quad\ \ \underline{\overset{-}{1} \overset{-}{+} x} \\
\qquad\quad -x \\
\qquad\quad \underline{\overset{+}{-}x \overset{+}{-} x^2} \\
\qquad\qquad\quad x^2 \\
\qquad\qquad\quad \underline{\overset{-}{x^2} \overset{-}{+} x^3} \\
\qquad\qquad\qquad -x^3
\end{array}
$$

Q. find the generating function.

$1, 0, 0, 1, 0, 0, 1, 0, 0, \ldots\ldots$

Soln. $a_0 = 1,\ a_1 = 0;\ a_2 = 0,\ a_3 = 1, \ldots$

We have

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdot$$

$$g(x) = 1 + x^3 + x^6 + \cdots$$

$$= \frac{1}{1-x^3}$$

$$1-x^3 \overline{)\, 1 \quad} (1+x^3$$

$$\frac{1-x^3}{\underline{-\quad +}}$$

$$x^3$$

$$\frac{x^3 - x^6}{\underline{-\quad +}}$$

$$x^6$$

Q. Find the generating function

i) $\{1, 1, 1, \ldots\}$

ii) $\{1, -2, 3, -4, \ldots\}$

iii) $\{0, 1, 2, 3.\}$

iv) $\{2, 5, 8, 11, 14, \ldots\}$

Sol^n. i) $\{1, 1, 1, \ldots\}$

$$g(x) = 1 + x + x^2 + \ldots$$

$$= (1-x)^{-1}$$

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ii) $\{1, -2, 3, 4\}$

$$g(x) = 1 - 2x + 3x^2$$
$$= (1+x)^{-2}$$

iii) $\{0, 1, 2, 3\}$

$$g(x) = \{0, 1, 2, 3\}$$
$$= 0 + 1\cdot x + 2x^2 + 3x^3 + \cdots$$
$$= x + 2x^2 + 3x^3 + \cdots$$
$$= x(1 + 2x + 3x^2 + \cdots)$$
$$= x(1-x)^{-2}$$

iv) $\{2, 5, 8, 11, 14, \ldots\}$

$$g(x) = 2 + 5x + 8x^2 + 11x^3 + 14x^4$$
$$= (2 + 2x + 2x^2 + 2x^3 + 2x^4)$$
$$+ (3x + 6x^2 + 9x^3 + 12x^4)$$

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$$= 2\left(1+x+x^2+x^3+x^4+\cdots\right)$$

$$+ 3x\left(1+2x+3x^2+4x^3\right)$$

$$= 2(1-x)^{-3} + 3x(1-x)^{-2}$$

Recurrence Relation:-

A recurrence relation is an eq^n that recursively define a sequence where term is function of previous term.

Ex (1) $a_1 = a_0 + 2$

$a_2 = a_1 + 2$

$a_3 = a_2 + 2$

- - - - - - -

- - - - - - - -

$a_n = a_{n-1} + 2$

(11) $a_n = 3 \cdot 2^n$

$a_n = a_{n-1} \cdot 3$

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Q. $a_k = a_{k-1} + 2$ with $a_0 = 1$

Sol^n. Put $k = 1, \ldots\ldots n$

$a_1 = a_0 + 2 = 1+2$

$a_2 = a_1 + 2 = 1+2+2 = 1+2.2$

$a_3 = a_2 + 2 = 1+2+2+2 = 1+3.2$

$- \quad - \quad - \quad - \quad - \quad - \quad -$

$a_n = 1 + n.2$

$\therefore \quad a_n = 1 + 2n.$

Iteration method.

Solving Recurrence Relation :-

There are 3 methods of solving

(1) Iteration

(2) characteristics roots

(3) Generating function.

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Type I :- First order linear recurrence relation.

Q. $a_{n+1} = 2a_n$, $a_0 = 1$

soln. The method is

Linear eqn

↓

Find root (r)

↓

$a_n = A \cdot (r^n)$. Find A.

04 Sunday Now, $a_{n+1} = 2a_n$, $a_0 = 1$

$$r^{n+1} = 2 \cdot r^n \quad —(1)$$

Divide by r^n in eqn (1), then

$$\frac{r^{n+1}}{r^n} = \frac{2 \cdot r^n}{r^n}$$

$$r = 2$$

$$a_n = A \cdot r^n$$

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when $n = 0$, then

$a_0 = A \cdot 2^0$

$1 = A$

substitute the value

$a_n = 1 \cdot 2^n$

Q. $a_n = 7 a_{n-1}$; $a_2 = 98$

Sol^n. $r^n = 7 \cdot r^{n-1}$

Divide by r^{n-1}

$$\frac{r^n}{r^{n-1}} = \frac{7 \cdot r^{n-1}}{r^{n-1}}$$

$\therefore$ $r = 7$.

$a_n = A \cdot r^n$

$a_2 = A \cdot 7^2$

$98 = A \cdot 49$

$\therefore$ $A = 2$

$a_n = 2 \cdot 7^n$.

Recurrence relation :- A recurrance relation of the sequence $\{a_n\}$ is an eq^n that expresses a_n in terms of one or more of the previous terms of the sequence, $a_0, a_1, a_2 \ldots a_n$ for all integers n with $n \geq n_0$, where n_0 is non-negative integer. It is also called difference eq^n.

Sol^n of rec. rel^n !
A sequence is called a solution of a rec. rel^n. of its terms satisfy the rec. rel^n.

Ex. Let $\{a_n\}$ be a sequence that satisfies that rec. rel^n $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, \ldots$ Suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Sol^n. $a_n = a_{n-1} - a_{n-2}$

Putting $n = 2, 3, \ldots$

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

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Recurrence relⁿ is in the form of

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_K a_{r-K} = f(r) \quad -(1)$$

this form is called Linear rec. relⁿ with constant coefficient.

where

$$C_0 \neq 0 \text{ and } C_K \neq 0$$

Second order rec. relⁿ.

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} = f(r)$$

Third order rec. relⁿ.

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + C_3 a_{r-3} = f(r)$$

Solⁿ of eqⁿ (1) is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

where $a_r^{(h)}$ = Homogeneous solⁿ.

$a_r^{(p)}$ = Particular solution.

Homogeneous Rec. relⁿ :- If $f(r) = 0$ then the eqⁿ(1) is called H.R.R.

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_K a_{r-K} = 0$$

Non-Homogeneous Rec. relⁿ:- If

$f(r) \neq 0$, then the eqⁿ(1)

is called Non-H. R. R.

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r)$$

Homogeneous linear Rec. Relⁿ with const. coeff.

Second order H. R. R. L is

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} = 0$$

Characteristic eqⁿ (Auxilliary Eqⁿ) is

$$C_0 m^2 + C_1 m + C_2 = 0$$

Case:- 1

If the roots of A.E are real and unequal

$m_1 \neq m_2$ ($m = 2, 3$)

General solⁿ is

$$a_r = C_1 m_1^r + C_2 m_2^r$$

Case :- 2

If the roots of A.E are real and equal

$m_1 = m_2 = m$, then

General sol^n is

$$a_r = (C_1 + rC_2)m^r$$

Case :- 3

If the roots of A.E are in complex Numbers.

$$m = \alpha \pm i\beta \quad [m = 2 \pm 3i]$$

General sol^n is

$$a_r = (C_1 \cos r\theta + C_2 \sin r\theta) R^r$$

where $R = \sqrt{\alpha^2 + \beta^2}$

and $\theta = \tan^{-1} \frac{\beta}{\alpha}$

Q. Solve the recurrence rel^n

$$a_r + 5a_{r-1} + 6a_{r-2} = 0$$

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Sol^n. Given

$$a_r + 5a_{r-1} + 6a_{r-2} = 0$$

which is second order rec. rel^n. —(1)

characteristic eq^n is (A.E)

$$m^2 + 5m + 6 = 0$$

$$m^2 + 2m + 3m + 6 = 0$$

$$m(m+2) + 3(m+2) = 0$$

$$(m+2)(m+3) = 0$$

$$m = -2, -3$$

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General sol^n is

$$a_r = C_1(-2)^r + C_2(-3)^r$$

Q. Solve the recurrence relation

$a_r - 7a_{r-1} + 10a_{r-2} = 0$ given that

$a_0 = 0,\ a_1 = 3$

Sol^n. Given rec. rel^n. is

$$a_r - 7a_{r-1} + 10a_{r-2} = 0 \quad -(1)$$

$$a_0 = 0,\ a_1 = 3$$

characteristics eq^n is

$$m^2 - 7m + 10 = 0$$

$$m^2 - 2m - 5m + 10 = 0.$$

$$m(m-2) - 5(m-2) = 0$$

$$(m-2)(m-5) = 0$$

$$m_{1,2} = 2, 5$$

h.s is

$$a_r = C_1(2)^r + C_2(5)^r \quad -(2)$$

Putting in eq^n (2), $a_0 = 0.$

$a_r = 0$

$r = 0$

$$C_1(2)^0 + C_2(5)^0 = 0$$

$$C_1 + C_2 = 0 \quad -(3)$$

Again Putting $a_1 = 3$, $a_r = 3$, $r = 1$

$$C_1(2)^1 + C_2(5)^1 = 3$$

$$2C_1 + 5C_2 = 3 \quad -(4)$$

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Solving (3) and (4), we get

$$c_1 = -1, \quad c_2 = 1$$

General solⁿ is

$$\underline{a_r = 5^r - 2^r}$$

Q. Solve the recurrence relation

$$a_r - 3a_{r-1} + 3a_{r-2} - a_{r-3} = 0$$

given that $a_0 = 1, a_1 = -2, a_2 = -1$

Solⁿ. Given

$$a_r - 3a_{r-1} + 3a_{r-2} - a_{r-3} = 0 \quad \text{---(1)}$$

Which is third order recu. relⁿ.

characteristic eqⁿ is

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$(m-1)^3 = 0$$

$$m = 1, 1, 1$$

General sol^n is

$$a_r = (c_1 + c_2 r + c_3 r^2)(1)^r$$

$$\therefore a_r = c_1 + c_2 r + c_3 r^2 \quad \text{---(2)}$$

Putting in eq^n (2),

$a_0 = 1$, $a_r = 1$ and $r = 0$

$c_1 = 1$.

Putting in eq^n (2),

$a_1 = -2$

$a_r = -2$

$r = 1$

$$c_1 + c_2(1) + c_3(1)^2 = -2$$

$$c_1 + c_2 + c_3 = -2 \quad \text{(3)}$$

Putting in eq^n (2), $a_2 = -1$

$a_r = -1$

$r = 2$

$$C_1 + C_2(2) + C_3(2)^2 = -1$$

$$C_1 + 2C_2 + 4C_3 = -1 \quad -(4)$$

Solving eq^n (3) and (4), we get

$C_2 = -11/2$ and $C_3 = 5/2$

General sol^n is

$$a_r = 1 - \frac{11r}{2} + \frac{5}{2}r^2$$

Q. Solve the recurrence relation is

$$a_r + 2a_{r-1} + 2a_{r-2} = 0$$

given that $a_0 = 0$, $a_1 = -1$

Sol^n. Given

$$a_r + 2a_{r-1} + 2a_{r-2} = 0 \quad -(1)$$

$a_0 = 0$, $a_1 = -1$

which is s.o.r.r.

characteristic eqⁿ is

$$m^2+2m+2=0$$

$$m=\frac{-2\pm\sqrt{2^2-4(1).2}}{2(1)}$$

$$m=\frac{-2\pm 2i}{2}=-1\pm i=\alpha\pm\beta i$$

$$\alpha=-1,\quad \beta=1$$

$$\therefore R=\sqrt{\alpha^2+\beta^2}=\sqrt{(-1)^2+1^2}=\sqrt{2}$$

$$\theta=\tan^{-1}\left(\frac{\beta}{\alpha}\right)=\tan^{-1}\left(\frac{-1}{1}\right)$$

$$\theta=\tan^{-1}(-1)$$

$$\theta=\pi-\tan^{-1}(1)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$$

h.s is

$$q_r=(C_1\cos r\theta+C_2\sin r\theta).R^r$$

$a_r = [C_1 \cos r\theta + C_2 \sin r\theta] \cdot R^r$

$a_r = \left[C_1 \cos \frac{3\pi}{4} r + C_2 \sin \frac{3\pi}{4} r\right] (\sqrt{2})^r$ ——(2)

putting in eq^n (2),

$a_0 = 0$

$a_r = 0$

$r = 0$

$[C_1 (1) + C_2 (0)] (\sqrt{2})^0 = 0$

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$\Rightarrow C_1 = 0$

putting in eq^n (2),

$a_1 = -1$

$a_r = -1$

$r = 1$. then

$\left[C_1 \cos \frac{3\pi}{4} + C_2 \sin \frac{3\pi}{4}\right] (\sqrt{2})^1 = -1$

$$\left[c_1\left(\frac{-1}{\sqrt{2}}\right) + c_2\left(\frac{1}{\sqrt{2}}\right)\right](\sqrt{2})^1 = -1$$

$$\left[0 + c_2\left(\frac{1}{\sqrt{2}}\right)\right](\sqrt{2})^1 = -1$$

$\because c_1 = 0$

$\therefore c_2 = -1$

Putting the value in (2)

$$a_r = \left[0 + (-1)\sin\left(\frac{3\pi}{4}r\right)\right](\sqrt{2})^r$$

$$a_r = -\sin\left(\frac{3\pi}{4}r\right)(\sqrt{2})^r$$

Solving these

1) $a_r - 6a_{r-1} + 8a_{r-2} = 0$

given that $a_0 = 3,\ a_1 = 2$

Ans. $a_r = 5.2^r - 2.4^r$

2) $2a_r - 5a_{r-1} + 2a_{r-2} = 0$

given $a_0 = 0,\ a_1 = 1$

Ans. $a_r = \left(-\frac{2}{3}\right)\left(\frac{1}{2}\right)^r + \left(\frac{2}{3}\right)\cdot 2^r$

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3) $a_r - a_{r-1} - a_{r-2} = 0$

Ans. $a_r = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$

4) $a_r + 3a_{r-1} + 3a_{r-2} + a_{r-3} = 0$

given $a_0 = 1$, $a_1 = -2$, $a_2 = -1$

Ans. $a_r = (1 + 3r - 2r^2)(-1)^r$

5) $a_r + 6a_{r-1} + 25a_{r-2} = 0$

Ans. $a_r = [C_1 \cos r\theta + C_2 \sin r\theta](5)^r$

$\theta = \tan^{-1}\left(-\frac{4}{3}\right)$

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Problems of Generating Function

Q. find generating function of the numeric function $a_r = \{1, 3, 9, 27\}$.

Soln. Given $a_r = \{1, 3, 9, 27, \ldots\}$

$a_0 = 1,\ a_1 = 3,\ a_2 = 9,\ a_3 = 27,\ a_4 = 81$

We have

$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$g(x) = 1 + 3x + 9x^2 + 27x^3 + \ldots$

$= 1 + 3x + 3^2 x^2 + 3^3 x^3 + \ldots$

$= 1 + 3x + (3x)^2 + (3x)^3 + \ldots$

$= (1 - 3x)^{-1}$

Operations on generating function:-

Let $A(x), B(x), C(x)$ be the generating function, then

i) Sum of $A(x)$ and $B(x)$

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if $C_r = a_r + b_r$ then

$$C(x) = A(x) + B(x) = \sum_{r=0}^{\infty} (a_r + b_r) z^r$$

2) Scalar multiplication of K in $A(x)$

if $b_r = K \cdot a_r$ then

$$B(x) = K \cdot A(x) = K \sum_{r=0}^{\infty} a_r x^r$$

3) Product of $A(x)$ and $B(x)$

if $C_r = a_r \cdot b_r$ then

$$C(x) = A(x) \cdot B(x) = C_r \cdot x^r$$

4) Multiplication of α^r with a_r

if $b_r = \alpha^r \cdot a_r$ then

$$B(x) = A(\alpha x) = \sum_{r=0}^{\infty} a_r (\alpha x)^r$$

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Important Result for generating f^n:

1) $a_r = 1$, $g(x) = 1 + x + x^2 + \cdots = (1-x)^{-1} = \frac{1}{1-x}$

2) $a_r = K$, $g(x) = K + Kx + Kx^2 + \cdots = K(1-x)^{-1} = \frac{K}{1-x}$

3) $a_r = \alpha^r$, $g(x) = 1 + \alpha x + \alpha^2 x^2 + \cdots = (1-\alpha x)^{-1}$

$$= \frac{1}{1-\alpha x}$$

4) $a_r = r$,

$$g(x) = x + 2x^2 + 3x^3 + \cdots +$$

$$= x(1-x)^{-2} = \frac{x}{(1-x)^2}$$

5) $a_r = r^2$

$$g(x) = x + 2^2 x^2 + 3^3 x^3 + \cdots$$

$$= x(x+1)(1-x)^{-3}$$

$$= \frac{x(1+x)}{(1-x)^3}$$

6) $a_r = r(r+1)$. then

$$g(x) = 1\cdot 2x + 2\cdot 3x^2 + 3\cdot 4x^3 + \cdots$$

$$= \frac{2x}{(1-x)^3}$$

Q. Determine the discrete numeric function corresponding to the generating function

$$A(x) = \frac{2}{1-4x^2}$$

Sol^n. Given generating function is

$$A(x) = \frac{2}{1-4x^2} = \frac{2}{1-(2x)^2}$$

$$A(x) = \frac{2}{(1-2x)(1+2x)}$$

$$\therefore A(x) = \frac{A}{1-2x} + \frac{B}{1+2x}$$

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Solving these by partial fraction

$$A(x) = \frac{1}{1-2x} + \frac{1}{1+2x}$$

$\therefore$ Numeric function is

$$a_r = 2^r + (-2)^r$$

$$\left[\because \frac{1}{1-\alpha x} = \alpha^r, \quad \frac{1}{1+\alpha x} = (-\alpha)^r \right]$$

Q. Determine the discrete numeric function corresponding the generating f^n.

$$A(z) = \frac{1}{5-6z+z^2}$$

Sol^n. Given generating function is

$$A(z) = \frac{1}{z^2-6z+5} = \frac{1}{z^2-5z-z+5}$$

$$A(z) = \frac{1}{z(z-5)-1(z-5)} = \frac{1}{(z-5)(z-1)}$$

$$A(z) = \frac{A}{z-5} + \frac{B}{z-1}$$

Solving these:

$$A(z) = \frac{\frac{-1}{4}}{z-5} + \frac{\frac{1}{4}}{z-1}$$

$$\boxed{\cancel{A(z)} = \cancel{\frac{-1}{20}(5-z)^{-1}} \rightarrow}$$

$$= \frac{-1}{4} \cdot \frac{1}{-5(1-\frac{z}{5})} - \frac{1}{4} \cdot \frac{1}{1-z}$$

$$= \frac{1}{20}\left(1-\frac{z}{5}\right)^{-1} - \frac{1}{4}(1-z)^{-1}$$

$$a_r = \frac{-1}{4}(1)^r + \frac{1}{20}\left(\frac{1}{5}\right)^r$$

Q. Determine the discrete numeric function corresponding the generating function

$$A(z) = \frac{7z^2}{(1-2z)(1+3z)}$$

Sol^n: Given generating function

$$A(z) = \frac{7z^2}{(1-2z)(1+3z)} = 7\left(\frac{z^2}{1+z-6z^2}\right)$$

$$A(z) = 7\left[\frac{-1}{6} + \frac{\frac{z}{6}+\frac{1}{6}}{1+z-6z^2}\right]$$

$$A(z) = \frac{-7}{6} + \frac{7}{6}\left[\frac{z+1}{(1-2z)(1+3z)}\right]$$

Solve by partial fraction

$$A(z) = \frac{-7}{6} + \frac{7}{6}\left[\frac{3}{5(1-2z)} + \frac{2}{5(1+3z)}\right]$$

$$A(z) = \frac{7}{6}\left[\frac{3}{5}(1-2z)^{-1} + \frac{2}{5}(1+3z)^{-1} - 1\right]$$

$$a_r = \begin{cases} \frac{7}{6}\left(\frac{3}{5} + \frac{2}{5} - 1\right), & r = 0 \\ \frac{7}{6}\left(\frac{3}{5}\cdot 2^r + \frac{2}{5}(-3)^r\right), & r \geq 1 \end{cases}$$

$$a_r = \begin{cases} 0 & r = 0 \\ \frac{7}{30}\left(3.2^r + 2(-3)^r\right) & ; r \geq 1 \end{cases}$$

Q. Determine the discrete numeric function corresponding the generating function

$$A(z) = \frac{z^5}{5 - 6z + z^2}$$

Soln. Given generating function

$$A(z) = \frac{z^5}{5-6z+z^2} = \frac{z^5}{(1-z)(5-z)}$$

$$A(z) = z^5 \left[\frac{1}{(1-z)(5-z)}\right]$$

Solve (1) by partial fraction

$$A(z) = z^5 \left[\frac{1}{4}\left(\frac{1}{1-z}\right) - \frac{1}{4}\left(\frac{1}{5-z}\right)\right]$$

$$A(z) = z^5 \left[\frac{1}{4}(1-z)^{-1} - \frac{1}{20}\left(1 - \frac{z}{5}\right)^{-1}\right]$$

$$A(z) = z^5 \left[\frac{1}{4}\left(1+z+z^2+\ldots\right) - \frac{1}{20}\left(1 + \frac{z}{5} + \frac{z^2}{5^2} + \ldots\right)\right]$$

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$$A(z) = \frac{1}{4}\left(z^5 + z^6 + \dots + z^r + \dots\right)$$

$$- \frac{1}{20}\left(z^5 + \frac{z^6}{5} + \dots\right)$$

$$\therefore a_r = \begin{cases} 0 & ; 0 \le r \le 4 \\ \frac{1}{4} - \frac{1}{20}\left(\frac{1}{5^{r-5}}\right) & ; r \geq 5 \end{cases}$$

$$a_r = \begin{cases} 0 & ; 0 \le r \le 4 \\ 1/4\left(1 - \frac{1}{5^{r-4}}\right) & : r \geq 5 \end{cases}$$

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Q: Find generating function

$a_n - 7a_{n-1} + 10a_{n-2} = 0$ for all — ①

$n \geq 1$, and $a_0 = 3$, $a_1 = 3$.

Sol^n. We know

$$g(n) = \sum_{n=0}^{\infty} a_n x^n$$

Multiply x^n in both sides in eq^n (1)

$$a_n x^n - 7 a_{n-1} x^n + 10 a_{n-2} x^n = 0$$

summation from $n = 2$ to ∞ in both sides

$$\sum_{n=2}^{\infty} a_n x^n - 7 \sum_{n=2}^{\infty} a_{n-1} x^n + 10 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\left(\sum_{n=0}^{\infty} a_n x^n - a_0 - a_1 x\right) - 7x \left(\sum_{n=1}^{\infty} a_{n-1} x^{n-1} - a_0\right)$$

$$+ 10 x^2 \left(\sum_{n=2}^{\infty} a_{n-2} x^{n-2}\right) = 0$$

$$(g(x) - a_0 - a_1 x) - 7x \left[g(x) - a_0\right)$$

$$+ 10 x^2 (g(x)) = 0$$

$$(g(x) - 3 - 3x) - 7x \left[g(x) - 3\right]$$

$$+ 10 x^2 \left[g(x)\right] = 0$$

$$g(x) \left[10x^2 - 7x + 1\right] = 3 - 21x + 3x$$

$$g(x) \left[10x^2 - 7x + 1\right] = 3 - 18x$$

$$g(x) = \frac{3-18x}{10x^2-7x+1}$$

$$g(x) = \frac{3-18x}{(5x-1)(2x-1)}$$

Now, $$\frac{3-18x}{(5x-1)(2x-1)} = \frac{A}{5x-1} + \frac{B}{2x-1}$$

Solve by Partial fraction

$A = 1,\ B = 4$. then

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$$\frac{A}{5x-1} + \frac{B}{2x-1} = \frac{3-18x}{(5x-1)(2x-1)}$$

$$\Rightarrow \frac{3-18x}{(5x-1)(2x-1)} = \frac{1}{5x-1} + \frac{-4}{2x-1}$$

$$= \frac{4}{(1-2x)} - \frac{1}{(1-5x)}$$

$$= 4(1-2x)^{-1} - (1-5x)^{-1}$$

Q. Solve the recurrence relation

$$s_n + 3s_{n-1} - 4s_{n-2} = 0, \; n \geq 2$$

$s_0 = 3, \quad s_1 = -2$

Using Generating function

Sol^n. Given

$$s_n + 3s_{n-1} - 4s_{n-2} = 0$$

Multiply x^n both sides

$$s_n x^n + 3s_{n-1}x^n - 4s_{n-2} \cdot x^n = 0 \cdot x^n$$

Summation from $n = 2$ to ∞

$$\sum_{n=2}^{\infty} s_n x^n + 3\sum_{n=2}^{\infty} s_{n-1}x^n - 4\sum_{n=2}^{\infty} s_{n-2}x^n = 0$$

$$(g(x) - s_0 - s_1 x)\, x^0 + 3\left(g(x) - s_0\right)x^1$$

$$-4\left[g(x)\right]x^2 = 0$$

$$g(x)\left[1 + 3x - 4x^2\right] - s_0 - s_1 x - 3s_0 x = 0$$

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Apply given initial values S_0 and S_1

$$g(x)(1+3x-4x^2) - 3 + 2x - 9x = 0$$

$$g(x)(1+3x-4x^2) = 3+7x$$

$$\therefore g(x) = \frac{3+7x}{1+3x-4x^2}$$

$$= \frac{3+7x}{1+4x-x-4x^2}$$

$$= \frac{3+7x}{(1-x)(1+4x)}$$

$$g(x) = \frac{3+7x}{(1-x)(1+4x)} = \frac{A}{1-x} + \frac{B}{1+4x}$$

Solve by partial fraction,

$A = 2$, $B = 1$, then

$$g(x) = \frac{2}{(1-x)} + \frac{1}{1+4x}$$

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$\therefore S_n = 2(1)^n + (-4)^n$

$$\frac{1}{1-ax} = a^n$$

Q. Use the method of generating function to the recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2} + 4^n, \; n \geq 2$$

$a_0 = 2$ and $a_1 = 8$.

Sol^n. Recurrence relation are

$$a_n - 4a_{n-1} + 4a_{n-2} = 4^n$$

Multiply x^n both sides

$$a_n x^n - 4 a_{n-1} x^n + 4 a_{n-2} x^n = 4^n . x^n$$

Taking summation up from 2 to ∞

$$\sum_{n=2}^{\infty} a_n x^n - 4\sum_{n=2}^{\infty} a_{n-1} x^n + 4\sum_{n=2}^{\infty} a_{n-2} x^n = \sum_{n=2}^{\infty} 4^n x^n$$

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$$\left(g(x) - a_0 - a_1 x\right) - 4\left(g(x) - a_0\right)x + 4g(x)x^2$$

$$= \frac{1}{1-4x} - 1 - 4x$$

$$g(x)\left[1 - 4x + 4x^2\right] - a_0 - a_1 x + 4a_0 x = \frac{1}{1-4x} - 1 - 4x$$

putting initial value $a_0 = 2, a_1 = 8$

$$g(x)\left[1 - 4x + 4x^2\right] - 2 - 8x + 8x = \frac{1}{1-4x} - 1 - 4x$$

$$g(x)\left[1 - 4x + 4x^2\right] = \frac{1}{1-4x} - 1 + 2 - 4x$$

$$= \frac{1}{1-4x} + 1 - 4x$$

$$g(x)\left(1-2x\right)^2 = \frac{1 + 1 - 4x - 4x + 16x^2}{1-4x}$$

$$g(x) = \frac{2 - 8x + 16x^2}{(1-4x)(1-2x)^2}$$

Apply partial fraction method

$$\frac{16x^2 - 8x + 2}{(1-4x)(1-2x)^2} = \frac{A}{1-4x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2}$$

Solve these, we get

$A = 4$, $B = 0$, $C = -2$

$\boxed{\text{~~General term } a_n \text{ is~~} \quad \text{~~} a_n \text{~~}}$

$$\therefore g(x) = \frac{4 \cdot 1}{1-4x} - \frac{2}{(1-2x)^2}$$

$$\therefore a_n = 4(4)^n - 2(n+1) \cdot 2^n$$

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Q. Solve the recurrence relation

$$a_{n+2} + 3a_{n+1} + 2a_n = 3^n, \; n \geq 0$$

given $a_0 = 1$

Sol^n. Given recurrence relation

$$a_{n+2} + 3a_{n+1} + 2a_n = 3^n$$

Multiply each term by x^n.

$$a_{n+2} \cdot x^n + 3a_{n+1} \cdot x^n + 2a_n \cdot x^n = 3^n \cdot x^n$$

Taking summation from 0 to ∞

$$\sum_{n=0}^{\infty} a_{n+2} x^n + \sum_{n=0}^{\infty} 3a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^n$$

$$= \sum_{n=0}^{\infty} 3^n \cdot x^n$$

$$\sum_{n=0}^{\infty} a_{n-(-2)} \cdot x^n + 3 \sum_{n=0}^{\infty} a_{n-(-1)} x^n$$

$$+ 2 \sum_{n=0}^{\infty} a_{n-0} x^n = \sum_{n=0}^{\infty} 3^n \cdot x^n$$

$[g(x) - a_0 - a_1 x]x^{-2} + 3[g(x) - a_0]x^{-1}$

$+ 2[g(x)]x^0 = \frac{1}{1-3x}$

Multiply by x^2

$[g(x) - a_0 - a_1 x] + 3[g(x) - a_0]x^1$

$+ 2\,g(x) \cdot x^2 = \frac{x^2}{1-3x}$

$g(x)[1 + 3x + 2x^2] - a_0 - a_1 x - 3a_0 x$

$= \frac{x^2}{1-3x}$

Apply initial value, $a_0 = 1$, $a_1 = 2$

$g(x)(1 + 3x + 2x^2) - 1 - 2x - 3x = \frac{x^2}{1-3x}$

$g(x)(1 + 3x + 2x^2) = \frac{x^2}{1-3x} + 1 + 5x$

$= \frac{x^2 + 1 - 3x + 5x - 15x^2}{1-3x}$

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$g(x)(1+x)(1+2x) = \frac{-14x^2+2x+1}{1-3x}$

$\therefore g(x) = \frac{-14x^2+2x+1}{(1-3x)(1+x)(1+2x)}$

$g(x) = \frac{A}{1-3x} + \frac{B}{1+x} + \frac{C}{1+2x}$

Solve by partial fraction

$B = 5, \quad C = -\frac{14}{5}, \quad A = \frac{21}{20}$

$\therefore g(x) = \frac{1}{20} \cdot \frac{1}{1-3x} + 5 \cdot \frac{1}{1+x} - \frac{14}{5} \cdot \frac{1}{1+2x}$

General term

$\therefore a_n = \frac{1}{20}(3)^n + 5(-1)^n - \frac{14}{5}(-2)^n$

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Non-Homogeneous L.R.R. (Inhomogeneous R.R)

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + c_3 a_{r-3} + \cdots = f(r) \quad \text{--- (1)}$$

soln. of this R.R is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

where $a_r^{(h)}$ = Homogeneous soln

$a_r^{(p)}$ = Particular soln.

Method of finding the Particular Soln

when $f(r) \neq 0$ then

Case:- (1) when $f(r) = b^r$ and b is not a root of characteristic eqn.

We suppose by trial soln is

$$a_r = A \cdot b^r$$

Q: Solve the recurrence relation

$$a_r - 5a_{r-1} + 6a_{r-2} = 5^r$$

Sol^n. Given R.R. is

$$a_r - 5a_{r-1} + 6a_{r-2} = 5^r \quad \text{—(1)}$$

which is second order R.R.

characteristics eq^n (A.E) is

$$m^2 - 5m + 6 = 0$$

$$\Rightarrow \quad m = 2, 3.$$

Homogeneous eq^n is

$$a_r^{(h)} = C_1(2)^r + C_2(3)^r$$

15 Sunday $f(r) = 5^r$ (Non-Homo eq^n)

We assume the particular sol^n (P.S) is

$$a_r = A \cdot 5^r \quad \text{—(2)}$$

putting the value of a_r in eq^n(1)

$$A \cdot 5^r - 5(A \cdot 5^{r-1}) + 6(A \cdot 5^{r-2}) = 5^r$$

$$A \cdot 5^r - A \cdot 5^r + 6 \cdot A \cdot \frac{5^r}{5^2} = 5^r$$

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$$A \cdot 5^{r}\left(r - r + \frac{6}{25}\right) = 5^{r}$$

$$A = \frac{1}{\frac{6}{25}} = \frac{25}{6}$$

Putting the value of A in eqn (2)

$$a_r^{(P)} = \frac{25}{6} \cdot 5^{r} = \frac{1}{6} \cdot 5^{r+2}$$

complete soln. is

$$a_r = a_r^{(h)} + a_r^{(P)}$$

$$= C_1(2)^{r} + C_2(3)^{r} + \frac{1}{6} \cdot 5^{r+2}$$

Q. Solve the recurrence relation

$$a_r - 7a_{r-1} + 10a_{r-2} = 3^{r},$$

given that $a_0 = 0$ and $a_1 = 1$

soln. Given R.R. is

$$a_r - 7a_{r-1} + 10a_{r-2} = 3^{r} \quad -(1)$$

Characteristics eqn is

$$m^2 - 7m + 10 = 0$$

$\Rightarrow$ $m = 2, 5$ (Real and Unequal)

$$a_r^{(h)} = C_1(2)^r + C_2(5)^r$$

Particular soln is

$$a_r^{(p)} = A \cdot 3^r \quad \text{--- (2)}$$

putting the value of a_r in eqn (1)

$$A \cdot 3^r - 7(A \cdot 3^{r-1}) + 10(A \cdot 3^{r-2}) = 3^r$$

$$\Rightarrow A \cdot 3^r - \frac{7 \cdot A \cdot 3^r}{3} + \frac{10 \cdot A \cdot 3^r}{3^2} = 3^r$$

$$\Rightarrow A \cdot 3^r \left[1 - \frac{7}{3} + \frac{10}{9}\right] = 3^r$$

$$\Rightarrow A \cdot \cancel{3^r}\left(-\frac{2}{9}\right) = \cancel{3^r}$$

$$\therefore A = -\frac{9}{2}$$

Putting the value of A in eq^n (2)

$$a_r^P = \frac{-9}{2} \cdot (3^r) = -\frac{1}{2} \cdot 3^{r+2}$$

∴ C.S is

$$a_r = c_1 (2)^r + c_2 (5)^r - \frac{1}{2} \cdot 3^{r+2} \quad \longleftarrow (3)$$

Putting $a_0 = 0 \Rightarrow a_r = 0$ and $r = 0$.

from (3)

$$0 = c_1 (2)^0 + c_2 (5)^0 - \frac{1}{2} \cdot 3^2$$

$$\therefore c_1 + c_2 = \frac{9}{2} \longrightarrow (4)$$

Again putting $a_1 = 1 \Rightarrow a_r = 1 \Rightarrow r = 1$

from (3)

$$1 = c_1 (2)^1 + c_2 (5)^1 - \frac{1}{2} (3)^{2+1}$$

$$\therefore 1 = 2c_1 + 5c_2 - \frac{27}{2}$$

$$\therefore 2c_1 + 5c_2 = \frac{29}{2} \quad - (5)$$

Solving eqⁿ (4) and (5), we get

$$c_1 = \frac{8}{3}, \quad c_2 = \frac{11}{6}$$

putting the value of c_1 and c_2 in eqⁿ (3), we get the result.

$$a_r = \frac{8}{3}(2)^r + \frac{11}{6}(5)^r - \frac{1}{2} \cdot 3^{r+2}$$

Case :- (2). when $f(r) = b^r$ and b is a root of characteristics eqⁿ.

1) If multiplicity of b is one. then b is a root of multiplicity 1. then we suppose a trial solⁿ is

$$a_r = A \cdot r \cdot b^r$$

2) If multiplicity of b is two. b is root of multiplicity 2. then

$$a_r = A \cdot r^2 \cdot b^r$$

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3) If multiplicity of b is three. then

$$a_r = A \cdot r^3 \cdot b^r$$

Q. Solve the recurrence relation

$$a_r - 3a_{r-1} + 2a_{r-2} = 2^r$$

Soln. Given the R.R is

$$a_r - 3a_{r-1} + 2a_{r-2} = 2^r \quad \text{--- (1)}$$

characteristics eqn is

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow m = 1, 2$$

$$a_r^{(h)} = C_1 (1)^r + C_2 (2)^r$$

$\because$ $f(r) = 2^r$ and 2 is a characteristic root of multiplicity is 1, then the P.S is

$$a_r^{(P)} = A \cdot r \cdot 2^r \quad \text{--- (2)}$$

putting the value a_r in eqn (1)

$$A \cdot r \cdot 2^r - 3\left[A(r-1) \cdot 2^{r-1}\right] + 2\left[A(r-2) \cdot 2^{r-2}\right] = 2^r$$

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$$A\cdot r\cdot 2^{r} - \frac{3}{2}\cdot A r\cdot 2^{r} + \frac{3}{2}A\cdot 2^{r} + \frac{1}{2}A\cdot r\cdot 2^{r}$$

$$\therefore A\cdot 2^{r} = 2^{r}$$

$$A = 2$$

putting in eqn. (2), we get

$$a_r^{(P)} = 2\cdot r\cdot (2)^{r} = r\cdot 2^{r+1}$$

$\therefore$ C.S is

$$a_r = a_r^{(h)} + a_r^{(P)}$$

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$$a_r = C_1(1)^{r} + C_2(2)^{r} + r\cdot 2^{r+1}$$

Q. Solve the reccurrence relation

$$a_r - 4a_{r-1} + 4a_{r-2} = 2^{r}$$

Soln. Given R.R is

$$a_r - 4a_{r-1} + 4a_{r-2} = 2^{r} \quad -(1)$$

characteristics eqn is

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$\Rightarrow$ $m = 2, 2$ (Roots are real and equal)

$$\therefore a_r^{(h)} = (C_1 + C_2 . r)(2)^r$$

$\because$ $f(r) = 2^r$ and 2 is a characteristics roots of multiplicity is 2. then P.S

$$a_r^{(P)} = A . r^2 . 2^r \quad \text{--- (2)}$$

Putting the value of a_r in eqn (1)

$$A . r^2 . 2^r - 4\left[A (r-1)^2 . 2^{r-1}\right] + 4\left[A (r-2)^2 . 2^{r-2}\right] = 2^r$$

$$\Rightarrow A . r^2 . 2^r - 4\left[A(r^2 - 2r + 1) . 2^{r-1}\right] + 4\left[A[r^2 - 4r + 4)\right] = 2^r$$

$$2^r . A\left[\cancel{r^2} - 2r^2 + \cancel{4r} - 2 + \cancel{r^2} - \cancel{4r} + 4\right] = \cancel{2^r}$$

$$A . 2 = 1 \Rightarrow A = 1/2$$

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Putting the value of A in eq^n(1)

$$a_r^{(P)} = \frac{1}{2} \cdot r^2 \cdot 2^r = r^2 \cdot 2^{r-1}$$

$\therefore$ C.S is

$$a_r = a_r^{(h)} + a_r^{(P)}$$

$$a_r = (C_1 + C_2 r)(2)^r + r^2 \cdot 2^{r-1}$$

Solving these

1) $a_r - 4a_{r-1} + 3 \cdot a_{r-2} = 5^r$

Ans. $a_r = C_1 (1)^r + C_2 (3)^r + \frac{5^r}{8}$

2) $a_r - 2a_{r-1} + a_{r-2} = 2^r$, $a_0 = 2, a_1 = 1$

Ans. $a_r = 1 - 2r + 2^r$

3) $a_r + 5a_{r-1} + 6a_{r-2} = 42 \cdot 4^r$

given $a_2 = 278$, $a_3 = 962$

Ans. $a_r = (-2)^r + 2(-3)^r + 16 \cdot 4^r$

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Case:(3). When $f(r)$ is a polynomial in r.

1) If $f(r)$ is first degree polynomial in r then the trial sol^n is

$$a_r^{(P)} = A_0 + A_1 r$$

2) If $f(r)$ is second degree polnomial in r, then the trial sol^n is

$$a_r^{(P)} = A_0 + A_1 r + A_2 r^2$$

3) If $f(r)$ is third degree. then

$$a_r^{(P)} = A_0 + A_1 r + A_2 r^2 + A_3 r^3$$

Q: Solve the recurrence relation $a_r - 5a_{r-1} + 6a_{r-2} = 2 + r$, given $a_0 = 1, a_1 = 1$

Sol^n: Given recurrence relation is

$$a_r - 5a_{r-1} + 6a_{r-2} = 2 + r \quad \text{---(1)}$$

Characteristics eq^n is

$$m^2 - 5m + 6 = 0$$

$$m = 2, 3$$

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$$a_r^{(h)} = C_1(2)^r + C_2(3)^r$$

$\because$ $f(r) = 2+r$ is a polynomial of degree one.

$$a_r^{(p)} = A_0 + A_1 r \quad \text{--- (2)}$$

Putting the value of a_r in eq^n (1)

$$(A_0 + A_1 r) - 5(A_0 + A_1(r-1))$$
$$+ 6(A_0 + A_1(r-2)) = 2+r$$

$$\Rightarrow A_0 + A_1 r - 5A_0 - 5A_1 r + 5A_1 + 6A_0 + 6A_1 r$$
$$- 12A_1 = 2+r$$

$$\Rightarrow (2A_0 - 7A_1) + 2A_1 r = 2 + r$$

Equating the coeff of r^0 and r^1 both side

$$2A_0 - 7A_1 = 2 \quad \text{--- (3)}$$

and $\quad 2A_1 = 1 \Rightarrow A_1 = \frac{1}{2}$

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putting the value of A_1 in eqⁿ (3)

$$A_0 = \frac{11}{4}$$

putting the value of a_r in eqⁿ (2)

$$a_r^{(P)} = \frac{11}{4} + \frac{1}{2} \cdot r$$

CS is

$$a_r = a_r^{(h)} + a_r^{(P)}$$

$$a_r = C_1 (2)^r + C_2 (3)^r + \frac{11}{4} + \frac{1}{2} \cdot r \quad -(4)$$

putting $a_0 = 1 \Rightarrow a_r = 1$ and $r = 0$.

from (4)

$$C_1 (2)^0 + C_2 (3)^0 + \frac{11}{4} + 0 = 1$$

$$C_1 + C_2 = \frac{7}{4} \quad -(5)$$

putting $a_1 = 1 \Rightarrow a_r = 1$ and $r = 1$

from (4).

$$C_1 (2)^1 + C_2 (3)^1 + \frac{11}{4} + \frac{1}{2}(1) = 1$$

$$\Rightarrow \quad 2C_1 + 3C_2 = -\frac{9}{4} \quad -(6)$$

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Solving eq^n (5) and (6)

$$c_1 = -3 \text{ and } c_2 = \frac{5}{4}$$

$\therefore$ from (4)

$$a_r = -3(2)^r + \frac{5}{4}(3)^r + \frac{11}{4} + \frac{1}{2} \cdot r$$

Q. Solve the recurrence relation

$$a_r - 4a_{r-1} + 4a_{r-2} = (r+1)^2$$

Sol^n. Given R.R is

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$$a_r - 4a_{r-1} + 4a_{r-2} = (r+1)^2 \quad -(1)$$

characteristics eq^n is

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0 \Rightarrow m = 2, 2$$

$\therefore a_r^{(h)} = (c_1 + c_2 r) \cdot (2)^r$

$\therefore f(r) = (r+1)^2$ of degree two.

$\therefore a_r^{(P)} = A_0 + A_1(r) + A_2 r^2$ — (2)

Putting the value of a_r in eq^n(1)

$$[A_0 + A_1 r + A_2 r^2] - 4[A_0 + A_1(r-1) + A_2(r-1)^2]$$
$$+ 4[A_0 + A_1(r-2) + A_2(r-2)^2]$$
$$= (r+1)^2$$

$$\Rightarrow [A_0 + A_1 r + A_2 r^2] - 4[A_0 + A_1 r - A_1 + A_2 r^2 - 2A_2 r + A_2]$$
$$+ 4[A_0 + A_1 r - 2A_1 + A_2 r^2 - 4A_2 r + 4A_2]$$
$$= (r+1)^2$$

$$\Rightarrow (A_0 - 4A_1 + 12A_2) + (A_1 - 8A_2)r$$
$$+ A_2 r^2 = 1 + 2r + r^2$$

Equating the coeff of r^0, r^1, r^2 both sides

$$A_0 - 4A_1 + 12A_2 = 1 \quad \text{— (4)}$$
$$A_1 - 8A_2 = 2 \quad \text{— (5)}$$
$$A_2 = 1 \quad \text{— (5)}$$

Solving eq^n (4), (5) and (6)

$A_0 = 29, \quad A_1 = 10$

$\therefore$ C.S is

$a_r = a_r^{(h)} + a_r^{(P)}$

$a_r = (C_1 + C_2 r)(2)^r + r^2 + 10r + 29$

Q. Solve the recurrence relation

$a_r + 6a_{r-1} + 9a_{r-2} = 3$

given that $a_0 = 0$ and $a_1 = 1$

Sol^n. Given R.R is

$a_r + 6a_{r-1} + 9a_{r-2} = 3 \quad -(1)$

characteristics eq.^{n} is

$m^2 + 6m + 9 = 0$

$(m+3)^2 = 0$

$(m+3)(m+3) = 0 \Rightarrow m = -3, -3$

$\therefore a_r^{(h)} = (C_1 + C_2 r) \cdot (-3)^r$

$\because f(r) = 3$ is a constant. then

$a_r^{(P)} = A_0$ — (2)

putting the value of a_r in eq^n(1)

$A_0 + 6A_0 + 9A_0 = 3$

$A_0 = \frac{3}{16}$

putting the value of A_0 in eq^n (2)

$a_r^{(P)} = \frac{3}{16}$

C.S is

$a_r = (C_1 + C_2 r)(-3)^r + \frac{3}{16}$ —(3)

putting $a_0 = 0 \Rightarrow a_r = 0 \Rightarrow r = 0$ in eq^n(3)

$(C_1 + C_2 \cdot 0)(-3)^0 + \frac{3}{16} = 0$

$C_1 = \frac{-3}{16}$

Putting $a_1 = 1 \Rightarrow a_r = 1$ and $r = 1$

in eq^n(3)

$$(C_1 + C_2 \cdot 1)(-3)^1 + \frac{3}{16} = 1$$

$$\Rightarrow 3C_1 + C_2 = -\frac{13}{16} \quad \text{--- (4)}$$

Putting the value of C_1 in eq^n (4)

$$C_4 = -\frac{1}{12}$$

Putting the value in eq^n (3)

$$a_r = \left[-\frac{3}{16} - \frac{1}{12} r\right](-3)^r + \frac{3}{16}$$

Q. Solve the recurrence relation

$$a_r - 2a_{r-1} + a_{r-2} = 7$$

Sol^n. Given R.R is

$$a_r - 2a_{r-1} + a_{r-2} = 7 \quad \text{--- (1)}$$

characteristics eq^n is

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0 \Rightarrow m = 1, 1$$

$a_r^{(h)} = (c_1 + c_2 r)(1)^r = c_1 + c_2 r$

$\because f(r) = 7 = 7 \cdot (1)^0$ is constant and double characteristics not

$\therefore \quad a_r^{(p)} = A_0 r^2$ — (2)

putting the value of a_r in eqn (1)

$A_0 r^2 - 2A_0(r-1)^2 + A_0(r-2)^2 = 7$

$A_0 r^2 - 2A_0 r^2 + 4A_0 r - 2A_0 + A_0 r^2 - 4A_0 r + 4A_0 = 7$

$2A_0 = 7 \Rightarrow A_0 = \frac{7}{2}$

putting the value in eqn (2)

$a_r^{(p)} = \frac{7}{2} r^2$

$\therefore$ C.S is $a_r = a_r^{(h)} + a_r^{(p)}$

$\therefore \quad a_r = c_1 + c_2 r + \frac{7}{2} r^2$

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Solving these

1) $a_r - 5a_{r-1} + 6a_{r-2} = r(r-1)$

Ans. $a_r = C_1(2)^r + C_2(3)^r + \frac{23}{2} + 3r + \frac{r^2}{2}$

2) $a_r - 2a_{r-1} = 7r^2$

Ans. $a_r = C_1(2)^r - 7r^2 - 28r - 42$

3) $a_r - a_{r-1} = 7$

Ans. $a_r = C_1(1)^r + 7r$

4)

Case-(4):- when $f(r) = b^r \cdot \phi(r)$

where $\phi(r)$ = Polynomial. then

$$f(r) = b^r \cdot \phi(r)$$

- b is not root of C.E
  - $\deg(\phi(r)) = 1$: $a_r^{(p)} = b^r (A_0 + A_1 r)$
  - $\deg \phi(r) = 2$: $a_r^{(p)} = b^r (A_0 + A_1 r + A_2 r^2)$
- b is a root of C.E
  - $m(b) = 1$: $a_r^{(p)} = b^r \cdot r\phi(r)$
    - $\deg \phi(r) = 1$: $a_r^{p} = b^r \cdot r(A_0 + A_1 r)$
  - $m(b) = 2$: $a_r^{(p)} = b^r r^2 \phi(r)$
    - $\deg \phi(r) = 1$: $a_r^{p} = b^r \cdot r^2 (A_0 + A_1 r)$

Q. Solve the recurrence relation

$$a_r + a_{r-1} = 3r \cdot 2^r \quad \text{—(1)}$$

Sol^n. Given R.R is

$$a_r + a_{r-1} = 3r \cdot 2^r \quad \text{—(1)}$$

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C.E is:

$$m+1=0 \Rightarrow m=-1$$

$$a_r^{(h)} = C_1(-1)^r$$

$\because f(r) = 3r.2^r$ is not a root of C.E and degree 1.

$$a_r^{(P)} = 2^r(A_0 + A_1 r) \quad \text{---} (2)$$

Putting the value of a_r in eq^n(1)

$$2^r(A_0+A_1 r) + 2^{r-1}\{A_0 + A_1(r-1)\} = 3r.2^r$$

$$\Rightarrow \cancel{2^r}\left[A_0 + A_1 r + \frac{A_0}{2} + \frac{A_1 r}{2} - \frac{A_1}{2}\right] = 3r.\cancel{2^r}$$

$$\left(\frac{3}{2}A_0 - \frac{A_1}{2}\right) + \frac{3}{2}A_1 r = 3r + 0$$

Equating both sides

$$\frac{3}{2}A_0 - \frac{A_1}{2} = 0 \quad \text{---} (3)$$

and $\frac{3}{2}A_1 = 3 \Rightarrow A_1 = 2$

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putting the value of A_1 in eqn (3)

$$A_0 = \frac{2}{3}$$

Putting the value of A_0, A_1 in eqn (2)

$$a_r^{(P)} = 2^r\left(\frac{2}{3} + 2r\right)$$

$\therefore$ C.S. is

$$a_r = a_r^{(h)} + a_r^{(P)}$$

$$a_r = C_1(-1)^r + \left(\frac{2}{3} + 2r\right) \cdot 2^r$$

Q. Solve the recurrence relation

$$a_r - 4a_{r-1} + 4a_{r-2} = (r+1) \cdot 2^r$$

Soln:- Given R.R is

$$a_r - 4a_{r-1} + 4a_{r-2} = (r+1) \cdot 2^r \quad \text{--- (1)}$$

C.E is

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0 \Rightarrow m = 2, 2$$

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$a_r^{(h)} = (C_1 + C_2 r)\cdot(2)^r$

$\because f(r) = 2^r\cdot(r+1)$

2 is a double root of c.E. and $(r+1)$ is a polynomical of degree one.

$$a_r^{(P)} = 2^r\cdot r^2\left[A_0 + A_1 r\right] \quad \text{---(2)}$$

Putting the value of a_r in eq^n(1)

$$2^r\cdot r^2\left[A_0 + A_1 r\right] - 4\cdot 2^{r-1}(r-1)^2\left[A_0 + A_1(r-1)\right]$$

$$+ 4\cdot 2^{r-2}\cdot(r-2)^2\left[A_0 + A_1(r-2)\right] = (r+1)\cdot 2^r$$

$$\Rightarrow \cancel{2^r}\left[A_0 r^2 + A_1 r^3 - 2(r^2-2r+1)(A_0 + A_1 r - A_1) + (r^2-4r+4)(A_0 + A_1 r - 2A_1)\right] = (r+1)\cdot\cancel{2^r}$$

$$\Rightarrow A_0 r^2 + A_1 r^3 - 2A_0 r^2 - 2A_1 r^3 + 2A_1 r^2 + 4A_0 r$$

$$+ 4A_1 r^2 - 4A_1 r - 2A_0 - 2A_1 r + 2A_1$$

$$+ A_0 r^2 + A_1 r^3 - 2A_1 r^2 - 4A_0 r - 4A_1 r^2 + 8A_1 r + 4A_0$$

$$+ 4A_1 r - 8A_1 = r+1$$

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$6A_1 r^1 + (2A_0 - 6A_1) r^0 = r + 1.$

Equating the coeff. of r^0 and r^1

$6A_1 = 1 \Rightarrow A_1 = \frac{1}{6}$

and, $2A_0 - 6A_1 = 1.$

$2A_0 - 6 \times \frac{1}{6} = 1$

$2A_0 = 2 \Rightarrow A_0 = 1$

Putting the value of A_0 and A_1 in eq^n (2)

$a_r^{(P)} = 2^r \cdot r^2 \left[1 + \frac{r}{6}\right]$

$\therefore$ C.S is $a_r^{(h)} + a_r^{(P)}$

$a_r = (C_1 + C_2 r)(2)^r + 2^r \cdot r^2 \left(1 + \frac{r}{6}\right)$

Case-(5). when $f(r) = b^r + p(r)$

Q. Solve the R.R:

$a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r$

given $a_0 = 1, a_1 = 1$

Soln. Given R.R is

$$a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r \quad \text{—} (1)$$

C.E is

$$m^2 - 5m + 6 = 0$$

$$\Rightarrow \quad m = 2, 3$$

$$a_r^{(h)} = C_1(2)^r + C_2(3)^r$$

$\because$ $f(r) = 2^r + r$,

2 is a root of C.E with multiplicity 1

13 Sunday and r is a polynomial of degree 1.

$$a_r^{(P)} = A_0 \cdot 2^r \cdot r + A_1 + A_2 r \quad \text{—} (2)$$

Putting the value of a_r in eqn (1).

$$[A_0 \cdot 2^r \cdot r + A_1 + A_2 \cdot r] - 5[A_0 \cdot 2^{r-1}(r-1) + A_1 + A_2(r-1)]$$
$$+ 6[A_0 \cdot 2^{r-2}(r-2) + A_1 + A_2(r-2)] = 2^r + r$$

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$\Rightarrow A_0.2^r.r + A_1 + A_2 r - 5A_0 r.2^{r-1} + 5A_0.2^{r-1} - 5A_1$

$- 5A_2 r + 5A_2 + 6A_0 r.2^{r-2} - 12A_0.2^{r-2} + 6A_1 + 6A_2 r$

$- 12A_2 = 2^r + r$

$\Rightarrow A_0.2^r.r + A_1 + A_2 r - \frac{5}{2} A_0.r.2^r + \frac{5}{2} A_0.2^r - 5A_1$

$- 5A_2 r + 5A_2 + \frac{3}{2} A_0 r.2^r - 3A_0.2^r + 6A_1 + 6A_2 r$

$- 12A_2 = 2^r + r$

$\Rightarrow A_0.2^r.r \left[1 - \frac{5}{2} + \frac{3}{2}\right] + A_0.2^r \left[\frac{5}{2} - 3\right]$

$+ 2A_2 r + 2A_1 - 7A_2 = 2^r + r$

$\Rightarrow -\frac{1}{2} A_0.2^r + 2A_2 r + (2A_1 - 7A_2) = 2^r + r + 0$

Equating on both sides

$-\frac{1}{2} A_0 = 1 \Rightarrow A_0 = -2$

and $2A_2 = 1$ and $2A_1 - 7A_2 = 0$

$A_2 = \frac{1}{2}$ $\Rightarrow 2A_1 - 7\left(\frac{1}{2}\right) = 0$

$\Rightarrow A_1 = \frac{7}{4}$

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putting in eqn (2)

$$a_r^{(P)} = -2(2^r \cdot r) + \frac{1}{2}r + \frac{7}{4}$$

$$a_r^{(P)} = -r \cdot 2^{r+1} + \frac{1}{2}r + \frac{7}{4}$$

$\therefore$ C.S is

$$a_r = a_r^{(h)} + a_r^{(P)}$$

$$a_r = C_1(2)^r + C_2(3)^r - r \cdot 2^{r+1} + \frac{r}{2} + \frac{7}{4} \quad \text{---} (3)$$

putting $a_0 = 1 \Rightarrow a_r = 1$ and $r = 0$.

then Eqn (3)

$$C_1(2)^0 + C_2(3)^0 - 0 \cdot 2^1 + 0 + \frac{7}{4} = 1$$

$$C_1 + C_2 = -\frac{3}{4} \quad \text{---} (4)$$

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Putting $a_1 = 1 \Rightarrow a_r = 1$ and $r = 1$. then eq^n (3).

$$C_1(2)^1 + C_2(3)^1 - 2^2 + \frac{1}{2} + \frac{7}{4} = 1$$

$$2C_1 + 3C_2 = \frac{11}{4} \quad -(5)$$

Solving eq^n (4) and (5)

$$C_1 = -5 \text{ and } C_2 = \frac{17}{4}$$

Putting in eq^n (3),

$$a_r = -5.(2)^r + \left(\frac{17}{4}\right)(3)^r - r.2^{r+1} + \frac{r}{2} + \frac{7}{4}.$$

solve these R.R

1. $a_r - 4a_{r-1} + 4a_{r-2} = r^2.2^r$

Ans. $a_r = (C_1 + C_2 r)(2)^r + \left(\frac{5}{12} + \frac{r}{3} + \frac{r^2}{12}\right)$

2. $a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r^2 + r$

Ans. $a_r = -2^{r+4} + \frac{31}{4}.3^r - r.2^{r+1}$
$+ \frac{r^2}{2} + 4r + \frac{37}{4}$

Q. Solve recurrence relation by generating function

$a_r - 2a_{r-1} - 3a_{r-2} = 0$, $r \geq 2$, $a_0 = 3$, $a_1 = 1$

Soln. Multiplying bothsides z^r.

$a_r z^r - 2a_{r-1} z^r - 3a_{r-2} \cdot z^r = 0$

$\because r \geq 2$, summing for all r

$$\sum_{r=2}^{\infty} a_r z^r - 2\sum_{r=2}^{\infty} a_{r-1} z^r - 3\sum_{r=2}^{\infty} a_{r-2} z^r = 0$$

1st term, $\sum_{r=2}^{\infty} a_r z^r = a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$

$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$

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$A(z) - a_0 - a_1 z = a_2 z^2 + a_3 z^3 + \cdots$

2nd term

$$\sum_{r=2}^{\infty} a_{r-1} z^r = a_1 z^2 + a_2 z^3 + a_3 z^4 + \cdots$$

$[A(z) - a_0] z = a_1 z^2 + a_2 z^3 + \cdots$

3rd term, $\sum_{r=2}^{\infty} a_{r-2} z^r = a_0 z^2 + a_1 z^3 + \cdots$

$A(z) \cdot z^2 = a_0 z^2 + a_1 z^3$

now, $[A(z) - a_0 - a_1 z] - 2z[A(z) - a_0] - 3A(z) z^2 =$

$[A(z) - 3 - z] - 2z[A(z) - 3] - 3[A(z) \cdot z^2] = 0$

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$$A(z) - 3 - z - 2zA(z) + 6z - 3A(z)z^2 = 0$$

$$A(z) = \frac{3-5(z)}{1-2z-3z^2} = \frac{3-5(z)}{(1-3z)(1+z)}$$

$$A(z) = \frac{1}{1-3z} + \frac{2}{1+z}$$

$$a_r = \underset{h.f}{1 \cdot 3^r} + \underset{H.f}{2(-1)^r}$$

$$\therefore A(z) = \frac{1}{1-az}, \quad a^r$$

$$A(z) = \frac{1}{(1-z)^2}, \quad r+1$$

$$A(z) = \frac{1}{(1-az)^2}, \quad (r+1) \cdot a^r$$

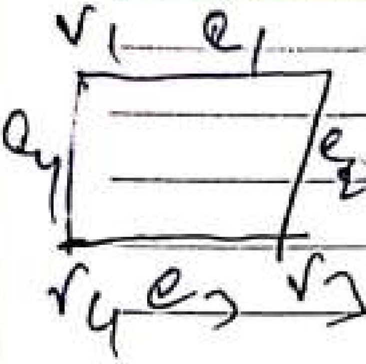
$$A(z) = \frac{z}{(1-z)^2}, \quad r$$

$$A(z) \frac{az}{(1-az)^2}, \quad ra^r$$

$$A(z) = e^z, \quad \frac{1}{r!}$$

$$A(z) = (1+z)^n, \quad \begin{cases} n!, & 0 \le r \le n \\ 0, & r > n \end{cases}$$

Graph: A graph G is a pair of sets (V, E).

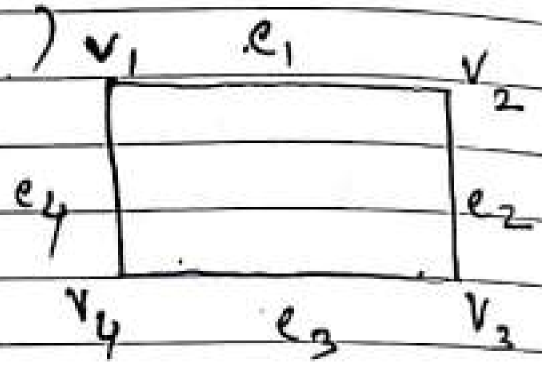


V: vertices (points, nodes)
E = edges (lines, arcs)

$V(G) = \{v_1, v_2, v_3, v_4\}$

$E(G) = \{e_1, e_2, e_3, e_4\}$

$G = (V, E)$



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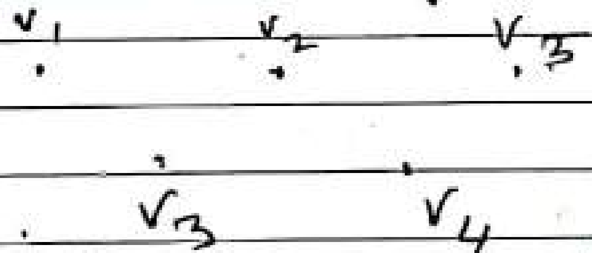
Null Graph: A graph in which number of edges = 0



Null Graph has 0 edge and 5 vertices.

Loop/self loop: An edge joining a vertex to itself is called loop. Connected twice.

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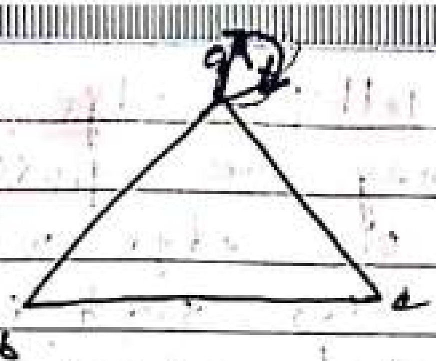
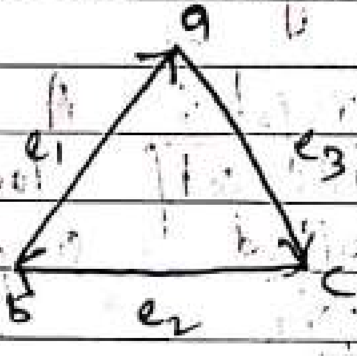


Directed Graph: The graph in which the elements of the edge set are ordered pair of vertices is called directed graph.

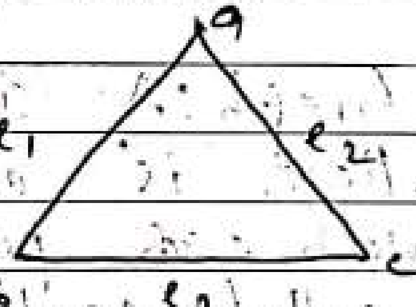
$e_1 = (b, a)$



$e_2 = (c, b)$

$e_3 = (a, c)$

Non Directed Graph: A graph in which the elements of edge set are unordered pair or of vertices is called a N.D.G.



$e_1 = \{a, b\}, \{b, a\}$

$e_2 = \{a, c\}, \{c, a\}$

$e_3 = \{b, c\}, \{c, b\}$