General Rules of differentiation

Theorem : If $\vec{u}(t)$ and $\vec{v}(t)$ be two differentiable functions of the scalar to show that

$$\frac{d}{dt}(\vec{u} \pm \vec{v}) = \frac{d\vec{u}}{dt} \pm \frac{d\vec{v}}{dt}$$

let $\vec{w} = \vec{u} \pm \vec{v}$ ——— (1)

let ∂u and ∂v be the very small increment in u and v respectively and $\partial\vec{w}$ be the corresponding small increment in $\vec{w}$

let $\vec{w} + \partial\vec{w} = \vec{u} + \partial\vec{u} \pm \vec{v} + \partial\vec{v}$ ——— (II)

Subtracting the corresponding sides of (1) from (2)

$$\vec{w} + \partial\vec{w} - \vec{w} = \vec{u} + \partial\vec{u} \pm \vec{v} + \partial\vec{v} - (\vec{u} \pm \vec{v})$$

$$\cancel{\vec{w}} + \partial\vec{w} - \cancel{\vec{w}} = \cancel{\vec{u}} + \partial\vec{u} \pm \cancel{\vec{v}} + \partial\vec{v} - \cancel{\vec{u}} \mp \cancel{\vec{v}}$$

$$\partial\vec{w} = \partial\vec{u} \pm \partial\vec{v}$$

Dividing both sides by ∂t

$$\frac{\partial\vec{w}}{\partial t} = \frac{\partial\vec{u} \pm \partial\vec{v}}{\partial t} = \frac{\partial\vec{u}}{\partial t} \pm \frac{\partial\vec{v}}{\partial t}$$

now taking Lt $\partial t \to 0$

$$\lim_{\partial t \to 0} \frac{\partial\vec{w}}{\partial t} = \lim_{\partial t \to 0} \frac{\partial\vec{u}}{\partial t} \pm \lim_{\partial t \to 0} \frac{\partial\vec{v}}{\partial t}$$

$$\frac{dw}{dt} = \frac{du}{dt} \pm \frac{dv}{dt}$$

Theorem II If $\vec{u}(t)$ be a differentiable vector function of the scalar t and $\varphi(t)$ the differentiable scalar function of t. To show that

$$\frac{d}{dt}(\vec{u}\,\varphi) = \frac{d\vec{u}}{dt}\varphi + \vec{u}\frac{d\varphi}{dt}$$

let $\vec{w} = \vec{u}\,\varphi$ ——— (1)

let $\partial\vec{u}$ and $\partial\varphi$ be the very small increments in $\vec{u}$ and φ respectively and $\partial\vec{w}$ be the corresponding small increment in $\vec{w}$

let $\vec{w} + \partial\vec{w} = (\vec{u} + \partial\vec{u})(\varphi + \partial\varphi)$

$$\vec{w} + \partial\vec{w} = \vec{u}\varphi + \vec{u}\partial\varphi + \varphi\partial\vec{u} + \partial\vec{u}\partial\varphi \quad \text{—(II)}$$

Subtracting the corresponding sides (I) from (II)

$$\vec{w} + \partial\vec{w} - \vec{w} = \cancel{\vec{u}\varphi} + \vec{u}\partial\varphi + \varphi\partial\vec{u} + \partial\vec{u}\partial\varphi - \cancel{\vec{u}\varphi}$$

$$\partial\vec{w} = \vec{u}\partial\varphi + \varphi\partial\vec{u} + \partial\vec{u}.\partial\varphi$$

Dividing both sides ∂t

$$\lim_{\partial t \to 0} \frac{\partial\vec{w}}{\partial t} = \frac{\vec{u}\partial\varphi}{\partial t} + \varphi\frac{\partial\vec{u}}{\partial t} + \frac{\partial\vec{u}.\partial\varphi}{\partial t}$$

now toking $\lim \partial t \to 0$

$$\lim_{\partial t \to 0} \frac{\partial\vec{w}}{\partial t} = \vec{u} \lim_{\partial t \to 0}\frac{\partial\varphi}{\partial t} + \varphi \lim_{\partial t \to 0}\frac{\partial\vec{u}}{\partial t} + \lim_{\partial t \to 0}\frac{\partial\vec{u}}{\partial t}.\partial\varphi$$

$$\frac{dw}{dt} = \vec{u}\frac{d\varphi}{dt} + \varphi\frac{du}{dt} + 0 \quad \{\partial\vec{u}.\partial\varphi \to 0\}$$

$$\frac{d(\vec{u}.\varphi)}{dt} = \vec{u}\frac{\partial\varphi}{\partial t} + \frac{d\vec{u}}{dt}\varphi$$

Theorem - 3 If $\vec{u}(t)$ and $\vec{v}(t)$ be two differentiable functions of the ~~scalar~~ vecto scalar t, to show that

$$\frac{d}{dt}(\vec{u}.\vec{v}) = \vec{u}\frac{d\vec{v}}{dt} + \vec{v}\frac{d\vec{u}}{dt}$$

let $\vec{w} = \vec{u}.\vec{v}$ ——(I)

let $\partial\vec{u}$ and $\partial\vec{v}$ be the very small increment in $\vec{u}$ and $\vec{v}$ respectively and ∂w be corresponding very small increment in w

$$w + \partial w = (\vec{u} + \partial\vec{u}).(\vec{v} + \partial\vec{v})$$

$$w + \partial w = \vec{u}.\vec{v} + \vec{u}\partial\vec{v} + \vec{v}\partial\vec{u} + \partial\vec{u}.\partial\vec{v} \text{ —— (II)}$$

Subtracting the corresponding sides of (I) from (2)

$$\cancel{w} + \partial w - \cancel{w} = \cancel{\vec{u}\vec{v}} + \vec{u}\partial\vec{v} + \vec{v}\partial\vec{u} + \partial\vec{u}.\partial\vec{v} - \cancel{\vec{u}\vec{v}}$$

$$\partial w = \vec{u}.\partial\vec{v} + \vec{v}\partial\vec{u} + \partial\vec{u}.\partial\vec{v}$$

Dividing both sides by ∂t

$$\frac{\partial w}{\partial t} = \frac{\vec{u}.\partial\vec{v}}{\partial t} + \frac{\vec{v}.\partial\vec{u}}{\partial t} + \frac{\partial\vec{u}}{\partial t}.\frac{\partial\vec{v}}{\partial t}.\partial t$$

now toking $\lim \partial t \to 0$

$\lim$

$$\lim_{\partial t\to 0}\frac{\partial w}{\partial t} = \vec{u}\cdot\lim_{\partial t\to 0}\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\lim_{\partial t\to 0}\frac{\partial \vec{u}}{\partial t} + 0$$

$$\frac{dw}{dt} = \vec{u}\cdot\frac{d\vec{v}}{dt} + \vec{v}\cdot\frac{d\vec{u}}{dt}$$

Theorem : 4

If vector $u(t)$ and vector $v(t)$ be two differential function of the scalar t to show that $\frac{d}{dt}(\vec{u}\times\vec{v}) = \vec{u}\times\frac{d\vec{v}}{dt} + \vec{v}\times\frac{d\vec{u}}{dt}$

let $\vec{w} = \vec{u}\times\vec{v}$ ——— (I)

$\vec{w}+\partial\vec{w} = (\vec{u}+\partial\vec{u})\times(\vec{v}+\partial\vec{v})$

(II) — (I)

$\cancel{\vec{w}} + \partial\vec{w} - \cancel{\vec{w}} = \cancel{\vec{u}\times\vec{v}} + \vec{u}\times\partial\vec{v} + \vec{v}\times\partial\vec{u} + \partial\vec{u}\times\partial\vec{v} - \cancel{\vec{u}\times\vec{v}}$

$\partial\vec{w} = \vec{u}\times\partial\vec{v} + \vec{v}\times\partial\vec{u} + \partial\vec{u}\times\partial\vec{v}$

$$\frac{\partial\vec{w}}{\partial t} = \vec{u}\times\frac{\partial\vec{v}}{\partial t} + \vec{v}\times\frac{\partial\vec{u}}{\partial t} + \frac{\partial\vec{u}}{\partial t}\cdot\frac{\partial\vec{v}}{\partial t}\cdot\partial t$$

now taking $\lim \partial t\to 0$

$$\lim_{\partial t\to 0}\frac{\partial\vec{w}}{\partial t} = \vec{u}\times\lim_{\partial t\to 0}\frac{\partial\vec{v}}{\partial t} + \vec{v}\times\lim_{\partial t\to 0}\frac{\partial\vec{u}}{\partial t} + 0$$

$$\frac{d\vec{w}}{dt} = \vec{u}\times\frac{d\vec{v}}{dt} + \vec{v}\times\frac{d\vec{u}}{dt}$$

$$\frac{d(\vec{u}\times\vec{v})}{dt} = \vec{u}\times\frac{d\vec{v}}{dt} + \vec{v}\times\frac{d\vec{u}}{dt}$$

Theorem – 5 Derivative of the scalar triple product

If $\vec{u}(t), \vec{v}(t), \vec{w}(t)$ be any three ~~vecto~~ vector functions of t then

$$\frac{d}{dt}[\vec{u}\cdot\vec{v}\ \vec{w}] = \left[\frac{d\vec{u}}{dt}\ \vec{v}\ \vec{w}\right] + \left[\vec{u}\ \frac{d\vec{v}}{dt}\ \vec{w}\right] + \left[\vec{u}\cdot\vec{v}\ \frac{d\vec{w}}{dt}\right]$$

We have $\frac{d}{dt}[\vec{u}\ \vec{v}\ \vec{w}] = \frac{d}{dt}\{\vec{u}\cdot(\vec{v}\times\vec{w})\}$

$$\frac{d\vec{u}}{dt}\cdot(\vec{v}\times\vec{w}) + \vec{u}\cdot\frac{d}{dt}(\vec{v}\times\vec{w})$$

$$= \left[\frac{d\vec{u}}{dt}\ \vec{v}\ \vec{w}\right] + \vec{u}\cdot\left\{\frac{d\vec{v}}{dt}\times\vec{w} + \vec{v}\times\frac{d\vec{w}}{dt}\right\}$$

$$= \left[\frac{d\vec{u}}{dt}\ \vec{v}\ \vec{w}\right] + \vec{u}\cdot\left(\frac{d\vec{v}}{dt}\times\vec{w}\right) + \vec{u}\cdot\left(\vec{v}\times\frac{d\vec{w}}{dt}\right)$$

$$= \left[\frac{d\vec{u}}{dt}\cdot\vec{v}\cdot\vec{w}\right] + \left[\vec{u}\ \frac{d\vec{v}}{dt}\cdot\vec{w}\right] + \left[\vec{u}\cdot\vec{v}\cdot\frac{d\vec{w}}{dt}\right]$$

proved.

Theorem – 6 Derivative of the vector triple product

If $\vec{u}(t), \vec{v}(t), \vec{w}(t)$ be any vector function of t then

$$\frac{d}{dt}\{\vec{u}\times(\vec{v}\times\vec{w})\} = \frac{d\vec{u}}{dt}\times(\vec{v}\times\vec{w}) + \vec{u}\times\left(\frac{d\vec{v}}{dt}\times\vec{w}\right) + \left[\vec{u}\times\vec{v}\times\frac{d\vec{w}}{dt}\right]$$

We have $\frac{d}{dt}\{\vec{u}\times(\vec{v}\times\vec{w})\}$

$$= \frac{d\vec{u}}{dt}\times(\vec{v}\times\vec{w}) + \vec{u}\times\frac{d\vec{v}}{dt}\times\vec{w} +$$

$$= \frac{d\vec{u}}{dt}\times(\vec{v}\times\vec{w}) + \vec{u}\left(\frac{d\vec{v}}{dt}\times\vec{w} + \vec{v}\times\frac{d\vec{w}}{dt}\right)$$

$$= \frac{d\vec{u}}{dt}\times(\vec{v}\times\vec{w}) + \vec{u}\times\left(\frac{d\vec{v}}{dt}\times\vec{w}\right) + \vec{u}\times\left(\vec{v}\times\frac{d\vec{w}}{dt}\right)$$

Q ① If $\vec{r} = \vec{i}\cos 2\pi t + 3\vec{j}\sin 2\pi t$ find

① $\frac{d\vec{r}}{dt}$, when $t=0$ ② $\frac{d^2\vec{r}}{dt^2}$ when $t=1$

③ $\left|\frac{d\vec{r}}{dt}\right|$, when $t=\frac{1}{6}$, and ④ $\left|\frac{d^2\vec{r}}{dt^2}\right|$, when $t=\frac{2}{3}$

We have $\vec{r} = \vec{i}\cos 2\pi t + 3\vec{j}\sin 2\pi t$

$$\frac{d\vec{r}}{dt} = \vec{i} \times -\sin 2\pi t \times 2\pi + 3\vec{j}\cos 2\pi t \times 2\pi$$

$$\frac{d\vec{r}}{dt} = -2\pi\vec{i}\sin 2\pi t + 6\pi\vec{j}\cos 2\pi t$$

$$\frac{d^2\vec{r}}{dt^2} = -2\pi i \times 2\pi\cos 2\pi t - 6\pi\vec{j} \times 2\pi\sin 2\pi t$$

$$\frac{d^2\vec{r}}{dt^2} = -4\pi^2\vec{i}\cos 2\pi t - 12\pi^2\vec{j}\sin 2\pi t$$

① At, $t=0$

$$\frac{d\vec{r}}{dt} = \vec{i} \times -\sin 2\pi \times 0 \times 2\pi + 6\pi\vec{j}\cos 2\pi \times 0$$

$$\frac{d\vec{r}}{dt} = 6\pi\vec{j} \text{ Ans.}$$

② At $t=1$

$$\frac{d^2\vec{r}}{dt^2} = -4\pi^2\vec{i}\cos 2\pi - 6\pi\vec{j} \times 2\pi\sin 2\pi$$
$$= -4\pi^2 i \times 1 = -4\pi^2\vec{i}$$

③ $\left|\frac{d\vec{r}}{dt}\right|$, $t=\frac{1}{6}$

$$\sqrt{\left(-2\pi\sin\frac{360}{6}\right)^2 + \left(6\pi\cos\frac{2\pi}{6}\right)^2}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4\pi^2 \times \frac{3}{4} + 36\pi^2 \times \frac{1}{4}} = \sqrt{3\pi^2 + 9\pi^2}$$
$$= \sqrt{12\pi^2} = \sqrt{4\pi^2 \times 3} = 2\pi\sqrt{3}$$

④ $\left|\frac{d^2\vec{r}}{dt^2}\right|$ at $t=\frac{2}{3}$

$$\frac{d^2\vec{r}}{dt^2} = -4\pi^2 i\cos\frac{4\pi}{3} - 12\pi^2\vec{j}\sin\frac{4\pi}{3} = 2\pi^2\vec{i} + 6\sqrt{3}\pi\vec{j}$$

$$\left|\frac{d^2r}{dt^2}\right| = \sqrt{(2\pi)^2 + (6\sqrt{3}\pi)^2} = 4\sqrt{7}\pi^2 \text{ Ans}$$

Q ② Find a unit tangent vector to any point on the space curve $x = a\cos t$, $y = a\sin t$, $z = bt$ where a and b are constant and t is the time

Solution The position vector $\vec{r}$ of any point (x, y, z) of the curve is given by

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + bt\,\vec{k}$$

Differentiating w.r. to x

$$\frac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + b\vec{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{a^2(\sin^2 t + \cos^2 t) + b^2} = \sqrt{a^2+b^2}$$

We know that unite vector

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} =$$

$$\frac{d\hat{r}}{dt} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{-a\cos t\,\vec{i} + a\cos t\,\vec{j} + b\vec{k}}{\sqrt{a^2+b^2}}$$ Ans

Q ③ If $\vec{a} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$, $\vec{b} = \vec{i}\cos t + \vec{j}\sin t$

$\vec{c} = 3t^2\vec{i} - 4t\vec{k}$ find $\frac{d}{dt}\{\vec{a}\times(\vec{b}\times\vec{c})\}$ when $t = \pi$

We know that vector triple product

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$$

$\vec{i}\cdot\vec{i} = 1$
$\vec{j}\cdot\vec{j} = 1$
$\vec{k}\cdot\vec{k} = 1$
$\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = 0$

$$\vec{a}\cdot\vec{c} = (t\vec{i} + t^2\vec{j} + t^3\vec{k})\cdot(3t^2\vec{i} - 4t\vec{k})$$
$$= 3t^3 - 4t^4$$

$$\vec{a}\cdot\vec{b} = (t\vec{i} + t^2\vec{j} + t^3\vec{k})\cdot(\vec{i}\cos t + \vec{j}\sin t)$$
$$= t\cos t + t^2\sin t$$

from formula

$$\frac{d}{dt}\{\vec{a}\times(\vec{b}\times\vec{c})\} = (\vec{a}\cdot\vec{c})\cdot\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$$
$$= (3t^3 - 4t^3)\cdot(\vec{i}\cos t + \vec{j}\sin t) -$$
$$(t\cos t + t^2\sin t)\cdot(3t^2\vec{i} - 4t\vec{k})$$

$=$

Q ③ If $\vec{r} = \sin t\,\hat{i} + \cos t\,\vec{j} + 1\vec{k}$ evaluate

① $\frac{d\vec{r}}{dt}$ ② $\frac{d^2\vec{r}}{dt^2}$ ③ $\left|\frac{dr}{dt}\right|$ ④ $\left|\frac{d^2r}{dt^2}\right|$

We have $\vec{r} = \sin t\,\hat{i} + \cos t\,\vec{j} + \vec{k}t$

$$\frac{d\vec{r}}{dt} = \cancel{\sin}\cos t\,\vec{i} - \sin t\,\vec{j} + \vec{k}$$

$$\frac{d^2\vec{r}}{dt^2} = -\sin t\,\hat{i} - \cos t\,\hat{j}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{1+1} = \sqrt{2}$$

$$\left|\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1$$

Q ④ Show that if $\vec{r} = \vec{a}\sin\omega t + b\cos\omega t$, where $\vec{a}$, $\vec{b}\,\omega$ are constant then

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r} \text{ and } \vec{r}\times\frac{d\vec{r}}{dt} = -\omega\vec{a}\times\vec{b}$$

We have $\vec{r} = \vec{a}\sin\omega t + \vec{b}\cos\omega t$

$$\frac{d\vec{r}}{dt} = \omega\vec{a}\cos\omega t - \omega\vec{b}\sin\omega t$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{a}\sin\omega t - \omega^2\vec{b}\cos\omega t$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2(\vec{a}\sin\omega t + \vec{b}\cos\omega t)$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$$

(b) $\vec{r}\times\frac{d\vec{r}}{dt} = (\vec{a}\sin\omega t + \vec{b}\cos\omega t)\times\cancel{(a\vec{\omega}\cos\omega t - \omega\vec{b}\sin\omega)}$
$\times(\omega\vec{a}\cos\omega t - \omega\vec{b}\sin\omega t)$

$= \omega\vec{a}\times\vec{a}\sin\omega t\cos\omega t - \vec{\omega}\,\vec{a}\times\vec{b}\sin\omega t\sin\omega t + \omega\vec{b}\times\vec{a}\cos\omega t\cdot\cos\omega t - \omega\vec{b}\times\vec{b}\cos\omega t\sin\omega t$

$= 0 - \omega\vec{a}\times\vec{b}\sin^2\omega t - \omega\vec{a}\times\vec{b}\cos^2\omega t - 0$

$= -\omega(\vec{a}\times\vec{b}\sin^2\omega t + \cancel{+}\,\cancel{\omega}\vec{a}\times\vec{b}\cos^2\omega t)$

$= -\omega\vec{a}\times\vec{b}(\sin^2\omega t + \cos^2\omega t) = \underline{-\omega\vec{a}\times\vec{b}}$

Q → ⑤ If $\vec{u} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$ and $\vec{v} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$ find $\frac{d}{dt}(\vec{u}\cdot\vec{v})$, when $t = 1$

Here given that

$$\vec{u} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$$

and $$\vec{v} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$$

$$\vec{u}\cdot\vec{v} = \left[t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}\right]\cdot\left[(2t-3)\hat{i} + \hat{j} - t\hat{k}\right]$$

$$\vec{u}\cdot\vec{v} = (2t-3)t^2 - t + (2t+1)\times -t]$$

$$\vec{u}\cdot\vec{v} = 2t^3 - 3t^2 - t - 2t^2 - t$$

$$\vec{u}\cdot\vec{v} = 2t^3 - 5t^2 - 2t$$

$$\frac{d}{dt}(\vec{u}\cdot\vec{v}) = \frac{d}{dt}(2t^3 - 5t^2 - 2t)$$

$$= 6t^2 - 10t - 2$$

$$\frac{d}{dt}(\vec{u}\cdot\vec{v})_{(t=1)} = 6\times 1 - 10\times 1 - 2$$

$$6 - 12 = -6 \text{ Ans}$$

Q → ⑥ If $\frac{d\vec{A}}{dt} = \vec{C}\times\vec{A}$ and $\frac{d\vec{B}}{dt} = \vec{C}\times\vec{B}$ prove that $\frac{d}{dt}(\vec{A}\times\vec{B}) = \vec{C}\times(\vec{A}\times\vec{B})$

Solution

$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \vec{A}\times\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\times\vec{B}$$

$$\vec{A}\times(\vec{C}\times\vec{B}) + (\vec{C}\times\vec{A})\times\vec{B}$$

$$(\vec{A}\cdot\vec{B})\vec{C} - (\vec{A}\cdot\vec{C})\vec{B} + (\vec{B}\cdot\vec{C})\vec{A} - (\vec{B}\cdot\vec{A})\vec{C}$$

$$\cancel{(\vec{A}\cdot\vec{B})\vec{C}} - (\vec{A}\cdot\vec{C})\vec{B} + (\vec{B}\cdot\vec{C})\vec{A} - \cancel{(\vec{A}\cdot\vec{B})\vec{C}}$$

$$= (\vec{B}\cdot\vec{C})\cdot\vec{A} - (\vec{A}\cdot\vec{C})\cdot\vec{B}$$

$$\vec{C}\times(\vec{A}\times\vec{B}) \text{ proved}$$

Q → (7) A particle moves along the curve $x = t^3+1$, $y = t^2$, $z = 2t+5$ where t is the time. Find the components of its velocity and acceleration at $t=1$ in the direction $\hat{i} - \hat{j} + 3\hat{k}$.

<u>Solution</u> let $\vec{r}$ be the position vector of the moving particle at time t.

$$\vec{r} = (t^3+1)\hat{i} + t^2\hat{j} + (2t+5)\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = 3t^2\hat{i} + 2t\hat{j} + 2\hat{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 6t\hat{i} + 2\hat{j}$$

$$\vec{v} = 3t^2\hat{i} + 2t\hat{j} + 2\hat{k}$$

$$\vec{a} = 6t\hat{i} + 2\hat{j}$$

$$\vec{P} = \hat{i} - \hat{j} + 3\hat{k}$$

$$\hat{p} = \frac{\vec{P}}{|\vec{P}|} = \frac{\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{1^2+1^2+3^2}} = \frac{\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{11}}$$

velocity in the direction of $\vec{p} = \vec{v}\cdot\hat{p}$

$$(3t^2\hat{i} + 2t\hat{j} + 2\hat{k}) \cdot \frac{\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{11}}$$

$$= \frac{3\cdot 1 + 2(-1) + 2\cdot 3}{\sqrt{11}} = \frac{3-2+6}{\sqrt{11}} = \frac{7}{\sqrt{11}} = \frac{7\sqrt{11}}{11}$$

and Acceleration in the direction of

$$\vec{p} = \vec{a}\cdot\hat{p} = (6i+2j) \cdot \frac{\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{11}}$$

$$\frac{6-2}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

For practice

① If $\vec{r} = 2e^{t}\hat{i} + \sin t\,\hat{j} + \log(1+t)\,\hat{k}$ find $\frac{d\vec{r}}{dt}$ and $\frac{d^2\vec{r}}{dt^2}$ at $t=0$

② If $\vec{r} = \cos mt\,\vec{i} + \sin mt\,\hat{j}$ where m is a constant and t variable show that $\vec{r} \times \frac{d\vec{r}}{dt} = m\hat{k}$

③ prove that $\frac{d}{dt}\left(\vec{a} \times \frac{d\vec{b}}{dt} - \frac{d\vec{a}}{dt} \times \vec{b}\right) = \vec{a} \times \frac{d^2\vec{b}}{dt^2} - \frac{d^2\vec{a}}{dt^2} \times \vec{b}$

④ $\vec{r} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$ find at $t=0$ the value of $\frac{d\vec{r}}{dt}$, $\frac{d^2\vec{r}}{dt^2}$, $\left|\frac{d\vec{r}}{dt}\right|$ and $\left|\frac{d^2\vec{r}}{dt^2}\right|$

⑤ prove that $\frac{d}{dt}[\vec{a}, \vec{b}, \vec{c}] = \left[\frac{d}{dt}\vec{a}, \vec{b}, \vec{c}\right] + \left[\vec{a}, \frac{d\vec{b}}{dt}, \vec{c}\right] + \left[\vec{a}, \vec{b}, \frac{d\vec{c}}{dt}\right]$

⑥ prove that $\frac{d}{dt}\{\vec{a} \times (\vec{b} \times \vec{c})\} = \frac{d\vec{a}}{dt} \times (\vec{b} \times \vec{c}) + \vec{a} \times \left(\frac{d\vec{b}}{dt} \times \vec{c}\right) + \vec{a} \times \left(\vec{b} \times \frac{d\vec{c}}{dt}\right)$

The vector differential operator ∇ (Del nabla) o

$$\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

This vector operator will be helpful in defining gradient, divergence and curl.

let the function $\varphi(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space. The co-ordinates x, y, z are increased by dx, dy, dz respectively

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz$$

$d\vec{r}$ the vector representing the displacement specified by dx, dy, dz then

$$d\vec{r} = \vec{i}dx + \vec{j}dy + \vec{k}dz$$

The gradient of φ, written as $\nabla\varphi$ or grad φ is defined by

$$\text{grad}\,\varphi \equiv \nabla\varphi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\varphi$$

$$= \vec{i}\frac{\partial \varphi}{\partial x} + \vec{j}\frac{\partial \varphi}{\partial y} + \vec{k}\frac{\partial \varphi}{\partial z}$$

= a vector quality

using Summation notation, $\text{grad}\,\varphi = \nabla\varphi = \sum \vec{i}\frac{\partial \varphi}{\partial x}$

Some directional derivative

1. $\nabla(u \pm v) = \nabla u \pm \nabla v$
2. $\nabla(au) = a\nabla u$ where a is constant
3. $\nabla(uv) = u\nabla v + v\nabla u$
4. $\nabla\left(\frac{u}{v}\right) = \frac{v\nabla u - u\nabla v}{v^2}$
5. $\nabla[f(u)] = f'(u)\nabla u$
6. $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$ where $\vec{a}$ is a constant vector.

Q ① Find grad φ when φ is given by

(i) $\varphi(x,y,z) = x^2y + xy^2 + z^2$ at the point $(1,1,1)$

(ii) $\varphi(x,y,z) = 3xz^2y - y^3z^2$ at the point $(1,-2,-1)$

① solution $\varphi(x,y,z) = x^2y + xy^2 + z^2$

By $\text{grad}\,\varphi = \vec{i}\frac{\partial \varphi}{\partial x} + \vec{j}\frac{\partial \varphi}{\partial y} + \vec{k}\frac{\partial \varphi}{\partial z}$

$= i\frac{\partial}{\partial x}(x^2y + xy^2 + z^2) + \vec{j}\frac{\partial}{\partial y}(x^2y + xy^2 + z^2) + \vec{k}\frac{\partial}{\partial z}(x^2y + xy^2 + z^2)$

$\vec{i}(2xy + y^2) + \vec{j}(x^2 + 2xy) + \vec{k}\,2z$

grad φ at $(1,1,1) = 3\vec{i} + 3\vec{j} + 2\vec{k}$

(ii) $\text{grad}\,\varphi = \nabla\varphi = \vec{i}\frac{\partial \varphi}{\partial x} + \vec{j}\frac{\partial \varphi}{\partial y} + \vec{k}\frac{\partial \varphi}{\partial z}$

$\vec{i}\frac{\partial}{\partial x}(3xyz^2 - y^3z^2) + \vec{j}\frac{\partial}{\partial y}(3xyz^2 - y^3z^2) + \vec{k}\frac{\partial}{\partial z}(3xyz^2 - y^3z^2)$

$= \vec{i}(3yz^2 - 0) + \vec{j}(3xz^2 - 3y^2z^2) + \vec{k}(6xyz - 2y^3z)$

grad φ at $(1,-2,-1)$

$= -6\vec{i} - 9\vec{j} - 4\vec{k}$

Q ② If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then show that

① $\text{grad}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$ ② $\nabla r^n = n r^{n-2}\vec{r}$

we know that for position vector

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r^2 = x^2 + y^2 + z^2$

Differentiating partially

$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x$

$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$\text{grad}\left(\frac{1}{r}\right) = \nabla\frac{1}{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$

$\text{grad}\left(\frac{1}{r}\right) = \vec{i}\frac{\partial \frac{1}{r}}{\partial x} + \vec{j}\frac{\partial \frac{1}{r}}{\partial y} + \vec{k}\frac{\partial \frac{1}{r}}{\partial z}$

$= \vec{i} - \frac{1}{r^2}\frac{\partial r}{\partial x} + \vec{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \vec{k} - \frac{1}{r^2}\frac{\partial r}{\partial z}$

$= -\frac{1}{r^2}\left(\vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\right)$

$-\frac{1}{r^2}\left(\frac{x\vec{i}}{r} + \frac{y\vec{j}}{r} + \frac{z\vec{k}}{r}\right)$

$$-\frac{1}{r^2} \times \frac{1}{r}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= -\frac{1}{r^3}\vec{r} = -\frac{\vec{r}}{r^3} \text{ Ans}$$

(2) Here given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial y}{\partial r} = \frac{y}{r}, \quad \frac{\partial z}{\partial r} = \frac{z}{r}$$

let $V = r^n$

$$\frac{\partial V}{\partial x} = n r^{n-1} \frac{\partial r}{\partial x}$$

$$\frac{\partial V}{\partial y} = n r^{n-1} \frac{\partial r}{\partial y}$$

$$\frac{\partial V}{\partial z} = n r^{n-1} \frac{\partial r}{\partial z}$$

$$\nabla V = \nabla r^n = \hat{i}\frac{\partial V}{\partial x} + \vec{j}\frac{\partial V}{\partial y} + \vec{k}\frac{\partial V}{\partial z}$$

$$\nabla V = \nabla r^n = \vec{i}\, n r^{n-1}\frac{\partial r}{\partial x} + \vec{j}\, n r^{n-1}\frac{\partial r}{\partial y} + n r^{n-1}\frac{\partial r}{\partial z}$$

$$\nabla r^n = n r^{n-1}\left\{\vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\right\}$$

$$\nabla r^n = \frac{n r^{n-1}}{r}\left\{x\vec{i} + y\vec{j} + z\vec{k}\right\}$$

$$= n r^{n-2}\vec{r}.$$

Q (3) If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $\omega = yz + xz + xy$ prove that $(\text{grad } u)[(\text{grad } v) \times (\text{grad } \omega)] = 0$

Solution :- $u = x + y + z$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial u}{\partial z} = 1$$

$$v = x^2 + y^2 + z^2$$

$$\frac{\partial v}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2y, \quad \frac{\partial v}{\partial z} = 2z$$

$$\frac{\partial \omega}{\partial x} = z + y, \quad \frac{\partial \omega}{\partial y} = z + x, \quad \frac{\partial \omega}{\partial z} = y + x$$

$$\text{grad } u = \nabla u = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)u$$

$$= \vec{i}\frac{\partial u}{\partial x} + \vec{j}\frac{\partial u}{\partial y} + \vec{k}\frac{\partial u}{\partial z}$$

$$= \vec{i}\frac{\partial(x+y+z)}{\partial x} + \vec{j}\frac{\partial(x+y+z)}{\partial y} + \vec{k}\frac{\partial(x+y+z)}{\partial z}$$

$$\text{grad } u = \nabla u = \vec{i}\frac{\partial x}{\partial x} + \vec{j}\frac{\partial y}{\partial y} + \vec{k}\frac{\partial z}{\partial z} = \vec{i} + \vec{j} + \vec{k}$$

$$\nabla v = \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2) + \vec{k}\frac{\partial (x^2+y^2+z^2)}{\partial z}$$

$$\text{grad } v = \nabla v = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\text{grad } w = \nabla w =$$

$$\vec{i}\frac{\partial}{\partial x}(yz+zx+xy) + \vec{j}\frac{\partial}{\partial y}(yz+zx+xy) + \vec{k}\frac{\partial}{\partial z}(yz+zx+xy)$$

$$\text{grad } w = \nabla w = \vec{i}(z+y) + \vec{j}(z+x) + \vec{k}(y+x)$$

Now $(\text{grad } u)[(\text{grad } v) \times \text{grad}(w)] =$

$$\nabla u \cdot (\nabla v \times \nabla w)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{vmatrix}$$

$R_3 \to R_3 + R_2$

$$2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$= 2(x+y+z)$ identical

$2(x+y+z) \times 0 = 0$

Q→1 Find the equation of the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$

Solution Let $\phi(x,y,z) = 2xz^2 - 3xy - 4x - 7$

$$\frac{\partial \phi}{\partial x} = 2z^2 - 3y - 4$$

$$\frac{\partial \phi}{\partial y} = -3x$$

$$\frac{\partial \phi}{\partial z} = 4xz$$

Now $\frac{\partial \phi}{\partial x}$ at the point $(1,-1,2) =$
$2z^2 - 3y - 4 = 8+3-4=7$

$\frac{\partial \phi}{\partial y}$ at the point $(1,-1,2)$
$-3x = 3\times -1 = -3$

$\frac{\partial \phi}{\partial z}$ at the point $(1,-1,2)$
$= 4xz = 4\times 1\times 2 = 8$

Equation of the plane at the point $(1,-1,2)$

$(x-1)\,7 + (y+1)\times -3 + (z-2)\times 8 = 0$

$7x - 7 - 3y - 3 + 8z - 16 = 0$

$7x - 3y + 8z - 26 = 0$

Q-2 Find ~~the~~ a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1,-1,2)$

Solution. The vector $\hat{n}$ normal to the surface $\phi = xy^3z^2 - 4$ is given by

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xy^3z^2 - 4)$$

$$\nabla\phi = i\,(y^3z^2) + j\,3xy^2z^2 + \vec{k}\,2xy^3z$$
$$= y^3z^2\vec{i} + 3xy^2z^2\vec{j} + 2xy^3z\vec{k}$$

At $(-1,-1,2)$

$$\nabla\phi = -4\vec{i} - 12\vec{j} + 4\vec{k}$$

$$|\nabla\phi| = \sqrt{(-4)^2 + (-12)^2 + (4)^2} = \sqrt{16+144+16} = \sqrt{4\times 4\times 11} = 4\sqrt{11}$$

$$\hat{n} \text{ at } (-1,-1,2) = \frac{-4\vec{i} - 12\vec{j} + 4\vec{k}}{4\sqrt{11}} = \frac{-\vec{i} - 3\vec{j} + \vec{k}}{\sqrt{11}}$$

Q → Find the angle between the surface $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$

let $\varphi_1 = x^2+y^2+z^2-9$ and $\varphi_2 = x^2+y^2-z-3$

$\nabla\varphi_1 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

At point $(2,-1,2)$ $\nabla\varphi_1 = 2\times2\vec{i} - 2\times1\times\vec{j} + 2\times2\vec{k}$

$= 4\vec{i} - 2\vec{j} + 4\vec{k}$

$\nabla\varphi_2 = 2x\vec{i} + 2y\vec{j} - \vec{k}$

At point $(2,-1,2)$: $\nabla\varphi_2 = 4\vec{i} - 2\vec{j} - \vec{k}$

let $\vec{n_1}$ and $\vec{n_2}$ be the vector along the normals to the surface φ_1 and φ_2 respectively at point $(1,-2,1)$

$\vec{n}_1 = 4\hat{i} - 2\hat{j} + 4\hat{k}$ and $\vec{n_2} = 4\hat{i} - 2\hat{j} - \hat{k}$

let θ be the angle between $\vec{n_1}$ and $\vec{n_2}$

$$\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \frac{(4\vec{i}-2\vec{j}+4\vec{k})\cdot(4\vec{i}-2\vec{j}-\vec{k})}{\sqrt{16+4+16}\sqrt{16+4+1}}$$

$$= \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$\therefore \cos\theta = \frac{8}{3\sqrt{21}}$

$\therefore \theta = \cos^{-1}\frac{8}{3\sqrt{21}}$

Q → Find the directional derivative of $\frac{1}{r}$ in the direction $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Here $\varphi(x,y,z) = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+$

$\frac{1}{r} = (x^2+y^2+z^2)^{-\frac{1}{2}}$

$$\nabla\varphi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$\nabla\varphi = \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}} + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-\frac{1}{2}} + \vec{k}(x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$= \vec{i}\left(-\tfrac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\times 2x\right] + \vec{j}\left[-\tfrac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\times 2y\right] + \vec{k}\left[-\tfrac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\times 2z\right]$$

$$\frac{2\vec{i}x + 2\vec{j}y + 2\vec{k}z}{2(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{\cancel{2}(x\vec{i}+y\vec{j}+z\vec{k})}{\cancel{2}(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$\hat{Q}$ - unit vector in the direction of

$$x\vec{i}+y\vec{j}+z\vec{k} = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{\sqrt{x^2+y^2+z^2}}$$

Directional derivative $= \nabla\varphi\cdot\hat{Q} =$

$$\frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{\frac{1}{2}}} \times \frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{\frac{1}{2}}}$$

$$= \frac{(x\hat{i}+y\hat{j}+z\hat{k})^2}{x^2+y^2+z^2}$$

Q → Find the directional derivative of the function $\varphi = 2xy + z^2$ at the point $(1,-1,3)$ in the direction of the vector $\hat{i}+2\hat{j}+2\hat{k}$

Here given $\varphi = 2xy + z^2$

$$\nabla\varphi = \vec{i}\frac{\partial}{\partial x}(2xy+z^2) + \vec{j}\frac{\partial}{\partial y}(2xy+z^2) + \vec{k}\frac{\partial}{\partial z}(2xy+z^2)$$

$$\nabla\varphi = 2y\vec{i} + 2x\vec{j} + 2z\vec{k}$$

At the point $(1,-1,3)$ $\nabla\varphi =$

$$-2\vec{i} + 2\vec{j} + 6\vec{k}$$

let $\vec{\alpha} = \vec{i} + 2\vec{j} + 2\vec{k}$

$$\hat{\alpha} = \frac{\vec{\alpha}}{|\alpha|} = \frac{\vec{i}+2\vec{j}+2\vec{k}}{\sqrt{9}}$$

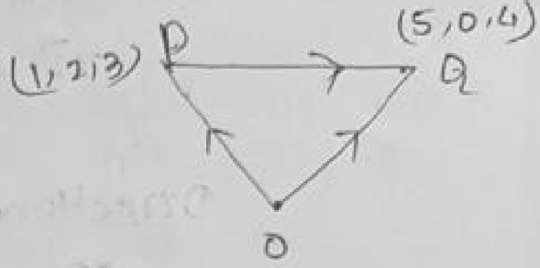
$$= \frac{1}{3}(\vec{i}+2\vec{j}+2\vec{k})$$

Directional derivative of φ in the direction $\vec{\alpha}$

$$\nabla\varphi\cdot\hat{\alpha} = (-2\hat{i}+2\hat{j}+6\hat{k})\cdot\frac{1}{3}(\hat{i}+2\hat{j}+2\hat{k})$$

$$= \frac{1}{3}(-2+4+12) = \frac{14}{3} \text{ Ans.}$$

Q → Find the directional derivative of the function $\varphi = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ where Q is the point (5, 0, 4)



$$\vec{OP} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{OQ} = 5\vec{i} + 0\vec{j} + 4\vec{k}$$

In ΔOPQ

$$\vec{OP} + \vec{PQ} = \vec{OQ}$$

$$\vec{PQ} = \vec{OQ} - \vec{OP} = 5\vec{i} + 0\vec{j} + 4\vec{k} - \vec{i} - 2\vec{j} - 3\vec{k}$$

~~Here $\varphi = x^2 - y^2 + 2z^2$~~ $= 4\vec{i} - 2\vec{j} + \vec{k}$

$$\nabla\varphi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\varphi$$

$$\varphi = x^2 - y^2 + 2z^2$$

$$\nabla\varphi = \vec{i}\frac{\partial(x^2 - y^2 + 2z^2)}{\partial x} + \vec{j}\frac{\partial}{\partial y}(x^2 - y^2 + 2z^2) + \vec{k}\frac{\partial}{\partial z}(x^2 - y^2 + 2z^2)$$

$$\nabla\varphi = 2x\vec{i} - 2y\vec{j} + 4z\vec{k}$$ at point $p(1,2,3)$

$$2\vec{i} - 4\vec{j} + 12\vec{k}$$

If $\hat{\alpha}$ is a unit vector in the direction $\vec{QP}$

$$\hat{\alpha} = \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{16+4+1}} = \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{21}}$$

Directional derivative of φ in the direction $\vec{QP}$

then $= \nabla\varphi\vec{\alpha}$

$$= (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{1}{\sqrt{21}}(4\vec{i} - 2\vec{j} + \vec{k})$$

$$\frac{8 + 8 + 12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

$$\frac{\overset{4}{28} \times \sqrt{21}}{21 \cdot 3} = \frac{4\sqrt{21}}{3}$$ Ans

Divergence of a vector point function :→

let $\vec{v}(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space. Then the divergence of $\vec{v}$ written as $\nabla\cdot\vec{v}$ or div $\vec{v}$ is defined by

$$\text{div}\,\vec{v} = \nabla\cdot\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\vec{v}$$

$$= \vec{i}\frac{\partial\vec{v}}{\partial x} + \vec{j}\frac{\partial\vec{v}}{\partial y} + \vec{k}\frac{\partial\vec{v}}{\partial z}$$

Using summation notation

$$\text{div}\,\vec{v} = \sum \vec{i}\frac{\partial\vec{v}}{\partial x}$$

let $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$

$$\nabla\cdot\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(v_1\vec{i} + v_2\vec{j} + v_3\vec{k})$$

$$\nabla\cdot\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

It is clearly div $\vec{v}$ is a scalar function

$\nabla\cdot\vec{v} = 0$, the vector field is called Solenoidal.

Curl of vector point function :-

let $\vec{v}(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space. Then the curl or rotation of $\vec{v}$, written as $\nabla\times\vec{v}$ or curl $\vec{v}$ or rot $\vec{v}$, is defined by

$$\text{curl}\,\vec{v} = \nabla\times\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\vec{v}$$

$$= \vec{i}\times\frac{\partial\vec{v}}{\partial x} + \vec{j}\times\frac{\partial v}{\partial y} + \vec{z}\times\frac{\partial v}{\partial z}$$

The curl of a vector point function is a vector quantity

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times (v_1\vec{i} + v_2\vec{j} + v_3\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_2 & v_3 \end{vmatrix} - \vec{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ v_1 & v_3 \end{vmatrix} + \vec{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ v_1 & v_2 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\vec{i} - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right)\vec{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial x}\right)\vec{k}$$

determinant the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ must precede v_1, v_2, v_3.

using summation notation ~~curl~~

$$\text{curl } \vec{V} = \Sigma\, \vec{i} \times \frac{\partial \vec{V}}{\partial x}$$

properties of the operator ∇

Theorem:- If φ and ψ be differential scalar functions of position (x, y, z) then to prove that

$$\nabla(\varphi \pm \psi) = \nabla\varphi \pm \nabla\psi$$

<u>Solution</u> we know that definition of

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\text{grad}(\varphi \pm \psi) = \nabla(\varphi \pm \psi)$$

$$\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\varphi \pm \psi)$$

$$= \vec{i}\frac{\partial}{\partial x}(\varphi \pm \psi) + \vec{j}\frac{\partial}{\partial y}(\varphi \pm \psi) + \vec{k}\frac{\partial}{\partial z}(\varphi \pm \psi)$$

$$= \vec{i}\frac{\partial \varphi}{\partial x} \pm \vec{i}\frac{\partial \psi}{\partial x} + \vec{j}\frac{\partial \varphi}{\partial y} \pm \vec{j}\frac{\partial \psi}{\partial y} + \vec{k}\frac{\partial \varphi}{\partial z} \pm \vec{k}\frac{\partial \psi}{\partial z}$$

$$=\left(\vec{i}\frac{\partial\varphi}{\partial x}+\vec{j}\frac{\partial\varphi}{\partial y}+\vec{k}\frac{\partial\varphi}{\partial z}\right)\pm\left(\vec{i}\frac{\partial\psi}{\partial x}+\vec{j}\frac{\partial\psi}{\partial y}+\vec{k}\frac{\partial\psi}{\partial z}\right)$$

$$=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\varphi\pm\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\psi$$

$$=\nabla\varphi\pm\nabla\psi$$

$$\text{grad}\,\varphi\pm\text{grad}\,\psi$$

Theorem :→ If $\vec{u}$ and $\vec{v}$ be defferentiable vector functions of position (x,y,z) to prove that

$$\nabla\cdot(\vec{u}\pm\vec{v})=\nabla\cdot\vec{u}\pm\nabla\cdot\vec{v}$$

$$\text{div}(\vec{u}\pm\vec{v})=\text{div}\,\vec{u}\pm\text{div}\,\vec{v}$$

Solution By definition $\nabla=\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}$

$$\text{div}(\vec{u}\pm\vec{v})$$

$$\nabla\cdot(\vec{u}\pm\vec{v})$$

$$=\cancel{\nabla\vec{u}}\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot(\vec{u}\pm\vec{v})$$

$$=\vec{i}\frac{\partial}{\partial x}(\vec{u}\pm\vec{v})+\vec{j}\frac{\partial}{\partial y}(\vec{u}\pm\vec{v})+\vec{k}\frac{\partial}{\partial z}(\vec{u}\pm\vec{v})$$

$$=\vec{i}\left(\frac{\partial\vec{u}}{\partial x}\pm\frac{\partial\vec{v}}{\partial x}\right)+\vec{j}\left(\frac{\partial u}{\partial y}\pm\frac{\partial v}{\partial y}\right)+$$

$$\vec{k}\left(\frac{\partial u}{\partial z}\pm\frac{\partial v}{\partial z}\right)$$

$$=\vec{i}\frac{\partial\vec{u}}{\partial x}+\vec{j}\frac{\partial u}{\partial y}+\vec{k}\frac{\partial u}{\partial z}\pm\left(\vec{i}\frac{\partial\vec{v}}{\partial x}+\vec{j}\frac{\partial\vec{v}}{\partial y}+\vec{k}\frac{\partial\vec{v}}{\partial z}\right)$$

$$=\left(i\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot\vec{u}\pm\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\vec{v}$$

$$=\nabla\cdot\vec{u}\pm\nabla\cdot\vec{v}$$

$$=\text{div}\cdot\vec{u}\pm\text{div}\cdot\vec{v}$$

To prove that

$$\nabla \times (\vec{u} \pm \vec{v}) = \nabla \times \vec{u} \pm \nabla \times \vec{v}$$

OR

$$\text{curl}(\vec{u} \pm \vec{v}) = \text{curl}\,\vec{u} \pm \text{curl}\,\vec{v}$$

<u>proof</u> By definition $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

Then $\text{curl} \times (\vec{u} \pm \vec{v})$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times (\vec{u} \pm \vec{v})$$

$$= \vec{i} \times \frac{\partial}{\partial x}(\vec{u} \pm \vec{v}) + \vec{j} \times \frac{\partial}{\partial y}(\vec{u} \pm \vec{v}) + \vec{k} \times \frac{\partial}{\partial z}(\vec{u} \pm \vec{v})$$

$$= \vec{i} \times \left(\frac{\partial \vec{u}}{\partial x} \pm \frac{\partial \vec{v}}{\partial x}\right) + \vec{j} \times \left(\frac{\partial \vec{u}}{\partial y} \pm \frac{\partial v}{\partial y}\right) + \vec{k} \times \left(\frac{\partial \vec{u}}{\partial z} \pm \frac{\partial \vec{v}}{\partial z}\right)$$

$$= \vec{i} \times \frac{\partial \vec{u}}{\partial x} + \vec{j}\frac{\partial u}{\partial y} + \vec{k}\frac{\partial u}{\partial z} \pm \left(\vec{i} \times \frac{\partial v}{\partial x} + \vec{j} \times \frac{\partial v}{\partial y} + \vec{k} \times \frac{\partial v}{\partial z}\right)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{u} \pm \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{v}$$

$$= \nabla \times \vec{u} \pm \nabla \times \vec{v}$$

$$= \text{curl}\,\vec{u} \pm \text{curl}\,\vec{v} \quad \text{prove}$$

To prove that $\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi$

or

$$\text{grad}(\varphi\psi) = \varphi\,\text{grad}\,\psi + \psi\,\text{grad}\,\varphi$$

<u>Solution</u> By definition $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

Then $\text{grad}(\varphi\psi) = \nabla(\varphi\psi)$

$$\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\varphi\psi)$$

$$= \vec{i}\frac{\partial}{\partial x}(\varphi\psi) + \vec{j}\frac{\partial}{\partial y}(\varphi\psi) + \vec{k}\frac{\partial}{\partial z}(\varphi\psi)$$

$$= \vec{i}\left\{\varphi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\varphi}{\partial x}\right\} + \vec{j}\left\{\varphi\frac{\partial\psi}{\partial y} + \psi\frac{\partial\varphi}{\partial y}\right\} + \vec{k}\left\{\varphi\frac{\partial\psi}{\partial z} + \psi\frac{\partial\varphi}{\partial z}\right\}$$

$$= \varphi\left\{\vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\psi}{\partial z}\right\} + \psi\left\{\vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}\right\}$$

$$= \varphi\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\psi + \psi\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\varphi$$

$$= \varphi\nabla\psi + \psi\nabla\varphi$$

$$= \varphi\,\text{grad}\,\psi + \psi\,\text{grad}\,\varphi$$

Theorem :- To prove that

$$\nabla\left(\frac{\varphi_1}{\varphi_2}\right) = \frac{\varphi_2 \nabla\varphi_1 - \varphi_1 \nabla\varphi_2}{\varphi_2^2}$$

OR

$$\text{grad}\left(\frac{\varphi_1}{\varphi_2}\right) = \frac{\varphi_2 \,\text{grad}\,\varphi_1 - \varphi_2 \,\text{grad}\,\varphi_2}{\varphi_2^2}$$

where φ_1 and φ_2 are two scalars parat

proof

$$\text{grad}\left(\frac{\varphi_1}{\varphi_2}\right) = \nabla\left(\frac{\varphi_1}{\varphi_2}\right)$$

$$\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{\varphi_1}{\varphi_2}\right)$$

$$= \vec{i}\frac{\partial}{\partial x}\left(\frac{\varphi_1}{\varphi_2}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{\varphi_1}{\varphi_2}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{\varphi_1}{\varphi_2}\right)$$

$$= \vec{i}\,\frac{\varphi_2\frac{\partial\varphi_1}{\partial x} - \varphi_1\frac{\partial\varphi_2}{\partial x}}{\varphi_2^2} + \vec{j}\,\frac{\varphi_2\frac{\partial\varphi_1}{\partial y} - \varphi_1\frac{\partial\varphi_2}{\partial y}}{\theta_2^2} + \vec{k}\,\frac{\varphi_2\frac{\partial\varphi_1}{\partial z} - \varphi_1\frac{\partial\varphi_2}{\partial z}}{\varphi_2^2}$$

$$= \frac{1}{\varphi_2^2}\left[\varphi_2\left(\vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}\right) - \varphi_1\left(\vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}\right)\right]$$

$$= \frac{\varphi_2\nabla\varphi_1 - \varphi_1\nabla\varphi_2}{\varphi_2^2}$$

(1) prove that $\text{grad}\ r^m = m r^{m-2}\vec{r}$

Solution $\text{Grad}\ r^m = \nabla r^m$

we know that $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

let $\varphi = r^m$

$$\frac{\partial\varphi}{\partial x} = m r^{m-1}\frac{\partial r}{\partial x}$$

$$\frac{\partial\varphi}{\partial y} = m r^{m-1}\frac{\partial r}{\partial y}$$

$$\frac{\partial\varphi}{\partial z} = m r^{m-1}\frac{\partial r}{\partial z}$$

$$\nabla r^m = \nabla\varphi = \left(\vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}\right)$$

$$= \vec{i}\,m r^{m-1}\frac{\partial r}{\partial x} + \vec{j}\,m r^{m-1}\frac{\partial r}{\partial y} + \vec{k}\,m r^{m-1}\frac{\partial r}{\partial z}$$

$$= m r^{m-1}\left(\vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\right)$$

$$= m r^{m-1}\left(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right)$$

$$= m r^{m-1}\cdot\frac{1}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$= m r^{m-2}\,\vec{r}$$

prove that $\operatorname{div}(r^n\vec{r}) = (n+3)r^n$

solution Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r},\quad \frac{\partial r}{\partial y} = \frac{y}{r},\quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

~~L.H.S div$(r^n\vec{r})$ = $\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(r^n$~~

L.H.S. $\operatorname{div}(r^n\vec{r})$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot r^n(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= n r^{n-1} x\frac{\partial r}{\partial x} + r^n\cdot 1 + n r^{n-1} y\frac{\partial r}{\partial y} + r^n\cdot 1 + n r^{n-1} z\frac{\partial r}{\partial z} + r^n\cdot 1$$

$$= 3r^n + n r^{n-1}\left\{x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right\}$$

$$= 3r^n + n r^{n-1}\left\{x\cdot\frac{x}{r} + \frac{y\cdot y}{r} + \frac{z\cdot z}{r}\right\}$$

$$= 3r^n + n r^{n-1}\frac{1}{r}(x^2 + y^2 + z^2)$$

$$= 3r^n + n r^{n-2}(x^2 + y^2 + z^2)$$

$$= 3r^n + n r^{n-2}\cdot r^2$$

$$3r^n + n r^n = (3+n)r^n$$

prove that div $(\vec{r} \times \vec{a}) = 0$ if a is a constant vector and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r^2 = x^2 + y^2 + z^2$

$\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

and $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

now $\vec{r} \times \vec{a} = (x\vec{i} + y\vec{j} + z\vec{k}) \times (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})$

$$= \begin{vmatrix} i & j & k \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$= (ya_3 - za_2)\vec{i} + (xa_3 - a_1z)\vec{j} + (xa_2 - a_1y)\vec{k}$

$\therefore$ div $(\vec{r} \times \vec{a})$

$= \nabla \cdot (\vec{r} \times \vec{a})$

$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left[(ya_3 - za_2)\vec{i} + (xa_3 - a_1z)\vec{j} + (xa_2 - a_1y)\vec{k}\right]$

$= \frac{\partial}{\partial x}(ya_3 - za_2) + \frac{\partial}{\partial y}(xa_3 - a_1z) + \frac{\partial}{\partial z}(xa_2 - a_1y)$

$= 0 + 0 + 0 = 0$

prove that $\nabla \cdot \vec{r} = 3$

let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\frac{\partial \vec{r}}{\partial x} = \vec{i}$, $\frac{\partial \vec{r}}{\partial y} = \vec{j}$, $\frac{\partial \vec{r}}{\partial z} = \vec{k}$

$\nabla \cdot \vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\vec{r}$

$= \vec{i}\frac{\partial \vec{r}}{\partial x} + \vec{j}\frac{\partial \vec{r}}{\partial y} + \vec{k}\frac{\partial \vec{r}}{\partial z}$

$= \vec{i}\cdot\vec{i} + \vec{j}\cdot\vec{j} + \vec{k}\cdot\vec{k}$

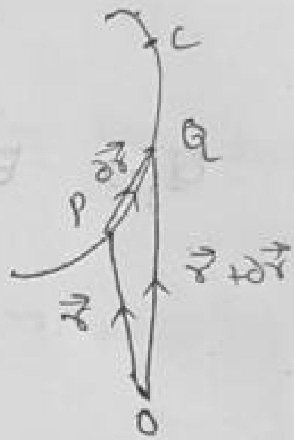
$= 1 + 1 + 1 = 3$

Line Integrals.

The locus of the point whose position vector $\vec{r}$ is a function of a scalar variable t is called a curve. The vector equation of a curve is

$$\vec{r} = \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

The curve is closed.



The differential displacement vector along a curve C at the point $P(\vec{r})$ is $d\vec{r}$.

So the displacement vector along the curve.

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

There are three types of line integral along a curve c.

① $\int_C f\, d\vec{r}$, where f is a scalar function of x, y, z.

② $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}$ is a vector function

③ $\int_C \vec{F} \times d\vec{r}$, when $\vec{F}$ is a vector function

① $$\int_C f\, d\vec{r} = \int_C f(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$

$$= \vec{i}\int_C f\,dx + \vec{j}\int_C f\,dy + \vec{k}\int_C f\,dz$$

② $$\int_C \vec{F}\cdot d\vec{r} = \int_C (F_1\vec{i} + F_2\vec{j} + F_3\vec{k})\cdot(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$

$$= \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

(iii) $\int_C \vec{F} \times d\vec{r} = \int_C (F_1\vec{i} + F_2\vec{j} + F_3\vec{k}) \times (dx\vec{i} + dy\vec{j} + dz\vec{k})$

$= \int_C \{ (F_2 dz - F_2 dy)\vec{i} + (F_3 dx - F_1 dz)\vec{j} + (F_1 dy - F_2 dx)\vec{k} \}$

Tangential line integral formula

$$\int_C \vec{F}\cdot d\vec{r} \quad \text{or} \quad \oint_C \vec{F}\cdot d\vec{r}$$

Q. → Evaluate the tangential line integral $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ and C is the curve $\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ where $-1 \leq t \leq 1$

<u>Solution</u> $\vec{F}\cdot d\vec{r} = (z\vec{i} + x\vec{j} + y\vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$

$= z\,dx + x\,dy + y\,dz$ —— (1)

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$

$x = t,\ y = t^2,\ z = t^3$

$dx = dt \quad dy = 2t\,dt \quad dz = 3t^2 dt$

$\int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1} t^3 dt + t\cdot 2t\,dt + t^2\, 3t^2 dt$

$= \int_{-1}^{1} \{ \int t^3 dt + \int 2t^2 dt + \int 3t^4 dt \}$

$= \left[\frac{t^4}{4} + 2\frac{t^3}{3} + \frac{3t^5}{5} \right]_{-1}^{1}$

$= \frac{1}{4} + \frac{2}{3} + \frac{3}{5} - \{ \frac{1}{4} - \frac{2}{3} - \frac{3}{5} \}$

$= \frac{1}{4} - \frac{1}{4} + \frac{2}{3} + \frac{2}{3} + \frac{3}{5} + \frac{3}{5}$

$0 + \frac{4}{3} + \frac{6}{5} = \frac{20+18}{15} = \frac{38}{15}$

Q→2 If a force $\vec{F} = 2x^2y\vec{i} + 3xy\vec{j}$ displaces a particle in the xy-plane from (0,0) to (1,4) along a curve $y = 4x^2$. Find the work done.

Solution

Here position vector is in the plane xy then z is must be zero.

$\vec{r} = x\vec{i} + y\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j}$

We know that work done by tangential integral $= \int_C \vec{F} \cdot d\vec{r}$

$= \int_C (2x^2y\vec{i} + 3xy\vec{j}) \cdot (dx\vec{i} + dy\vec{j})$

$= \int_C (2x^2y\,dx + 3xy\,dy)$ —— ①

Here given that $y = 4x^2$

$dy = 8x\,dx$

putting the value of y and dy in equ ①

$= \int_0^1 \{2x^2 \cdot 4x^2\,dx + 3x \cdot 4x^2 \cdot 8x\,dx\}$

$= \int_0^1 [8x^4\,dx + \int 96x^4\,dx] = \int_0^1 104x^4\,dx$

$= \left[\frac{104x^5}{5}\right]_0^1 = \frac{104}{5} - 0 = \frac{104}{5}$ ✓

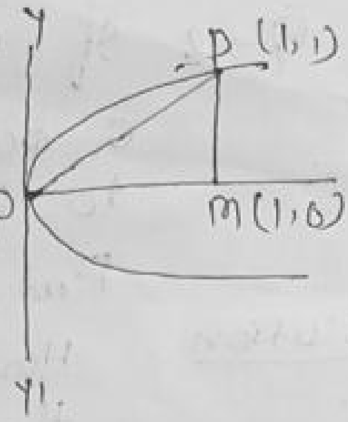
Q→3 Find $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = x^2\vec{i} + xy\vec{j}$ and c is the curve

① $y^2 = x$ joining (0,0) to (1,1)

② $y = x$ joining the same point.

③ Along omp

① $\int_C \vec{F} \cdot d\vec{r}$



$= \int_C (x^2 \vec{i} + xy \vec{j}) \cdot (dx\vec{i} + dy\vec{j})$

$= \int_C (x^2 dx + xy\, dy)$

consider the parabola path OP (0,0) to (1,1)

Equation of parabola $y^2 = x$

$2y\,dy = dx$

$\int_C (x^2 dx + xy\, dy)$

$\int_0^1 \left(\cancel{x^2} (y^2)^2 \cdot 2y\, dy + y^2 y\, dy \right)$

$= \int_0^1 (2y^5 dy + y^3 dx)$

$= \left[\frac{2y^6}{6} + \frac{y^4}{4} \right]_0^1 = \frac{\cancel{2}}{\cancel{6}_3} + \frac{1}{4} - 0 - 0$

$\frac{1}{3} + \frac{1}{4} = \frac{4+3}{12} = \frac{7}{12}$

② consider the straight line path $y = x$ Joining (0,0) to (1,1)

$y = x$

$dy = dx$

$\int_C x^2 dx + xy\,dx = \int_0^1 x^2 dx + x \cdot x\, dx$

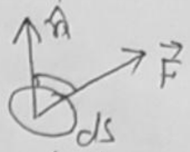
$= \int_0^1 (x^2 dx + x^2 dx) = 2\int_0^1 x^2 dx = x$

$2\left[\frac{x^3}{3} \right]_0^1 = \frac{2}{3}$ R

Surface Integral

$$\hat{n} = \frac{\text{grad} f}{|\text{grad} f|}, \quad ds = \frac{dx.dy}{|\hat{n}\hat{k}|}$$

$$|\hat{n}\hat{k}| = \frac{z}{a}$$



① A given surface s is projection on xy plane then

$$\iint_S \vec{F}\,\hat{n}\,ds = \iint_S \vec{F}.\hat{n}\,\frac{dx\,dy}{|\hat{n}k|}$$

$$|\hat{n}k| = \frac{z}{a}$$

② A given surface s is projection on yz plane. Then

$$\iint_S \vec{F}\,\hat{n}\,ds = \iint_S \vec{F}.\frac{dy\,dz}{|\hat{n}\,\hat{i}|}$$

$$|\hat{n}\hat{i}| = \frac{x}{b}$$

③ A given surface s is projction on the zx plane

$$\iint_S \vec{F}\,\hat{n}\,ds = \iint_S \frac{F\,dz\,dx}{|\hat{n}\,\hat{j}|}$$

$$|\hat{n}\,\hat{j}| = \frac{y}{c}.$$

Evaluate $\iint_S (yz\vec{i} + xz\vec{j} + xy\vec{z}).\vec{ds}$ where s is the surface of the sphere $x^2+y^2+z^2=a^2$ in the first oclant ~~quadrant~~.

Here $\varphi = x^2 + y^2 + z^2 - a^2$

vector normal to the surface

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla\varphi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-a^2)$$

$$\nabla\varphi = \cancel{2x}\,i\frac{\partial}{\partial x}(x^2+y^2+z^2-a^2) + j\frac{\partial}{\partial y}(x^2+y^2+z^2-a^2) +$$

$\vec{k}\, \frac{\partial}{\partial z}(x^2+y^2+z^2-a^2)$

$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{\sqrt{(2x)^2+(2y)^2+(2z)^2}}$

$\hat{n} = \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{2\sqrt{x^2+y^2+z^2}}$

$\hat{n} = \frac{\cancel{2}(x\vec{i}+y\vec{j}+z\vec{k})}{\cancel{2}\sqrt{a^2}} = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{a}$

Here $\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}$

$\vec{F}\hat{n} = (yz\vec{i}+xz\vec{j}+xy\vec{k})\frac{(x\vec{i}+y\vec{j}+z\vec{k})}{a}$

$\vec{F}\hat{n} = \frac{3\; xyz+xyz+xyz}{a} = \frac{3xyz}{a}$

now $\iint_S \vec{F}\hat{n}\,ds = \iint_S (\vec{F}\cdot\hat{n})\frac{dx\,dy}{|\hat{n}\,\vec{k}|}$

$\iint_S \frac{3xyz}{a}\frac{dx\,dy}{(\frac{z}{a})} =$

$\int_0^a\int_0^{\sqrt{a^2-x^2}} \frac{3xyz\,dx\,dy}{a(\frac{z}{a})}$

$\int_0^a\int_0^{\sqrt{a^2-x^2}} \frac{3xy\cancel{z}}{\cancel{z}}\,dx\,dy$

$= 3\left\{\int_0^a\int_0^{\sqrt{a^2-x^2}}\{xy\}\,dx\,dy\right.$

$= 3\left\{\int_0^a x\int_0^{\sqrt{a^2-x^2}} y\,dy\right\}$

$= 3\left\{\int_0^a \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx\right.$

$$\frac{3}{2}\int_0^a x\,dx\{\sqrt{a^2-x^2}\}^2 = \frac{3}{2}\int_0^a (a^2-x^2)x\,dx$$

$$= \frac{3}{2}\left[a^2\int x\,dx - \int x^3 dx\right]_0^a$$

$$= \frac{3}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{3}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{3}{2}\times\frac{a^4}{4} = \frac{3a^4}{4\times 2} = \frac{3a^4}{8}$$

Q→ Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$ and s is the surface of the plane $2x+3y+6z = 12$ in the 1st octant.

Here equation of the plane is $\phi \equiv 2x+3y+6z-12$

$$\text{grad}\,\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(2x+3y+6z-12)$$

$$\nabla\phi = 2\vec{i} + 3\vec{j} + 6\vec{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i}+3\vec{j}+6\vec{k}}{\sqrt{4+9+36}} = \frac{2\vec{i}+3\vec{j}+6\vec{k}}{7}$$

$$\vec{F}\cdot\hat{n} = (18z\vec{i} - 12\vec{j} + 3y\vec{k})\frac{(2\vec{i}+3\vec{j}+6\vec{k})}{7}$$

$$\frac{36z - 36 + 18y}{7} = \frac{18}{7}(2z-2+y)$$

$$\hat{n}\cdot\hat{k} = \frac{(2\vec{i}+3\vec{j}+6\vec{k})\cdot k}{7} = \frac{6}{7}$$

$$\int_S \vec{F}\cdot\hat{n}\,ds = \iint_R \frac{18}{7}(y+2z-2)\frac{7}{6}\,dx\,dy$$

$$3(y+2z-2)\,dx\,dy$$

$$\iint_R (3y+6z-6)\,dx\,dy$$

$$2x+3y+6z = 12$$

$$\iint_R (3y + 12 - 2x - 3y - 6)\,dx\,dy \qquad 6z = 12-2x-3y$$

$$2z = 4 - \frac{2x}{3} - y$$

$$\int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (6-2x)\, dy\, dx$$

$$= \int_0^6 \left[(6-2x) \int_0^{\frac{12-2x}{3}} dy \right] dx$$

$$= \int_0^6 \left[(6-2x) [y]_0^{\frac{12-2x}{3}} dx \right.$$

$$\int_0^6 \left[(6-2x) \left(\frac{12-2x}{3} \right) dx \right.$$

$$\frac{1}{3}\int_0^6 \left[72 - 12x - 24x + 4x^2 \right] dx$$

$$= \frac{1}{3}\int_0^6 \left[4x^2 - 36x + 72 \right] dx$$

$$= \cancel{\frac{4}{3}\int_0^6 \left[x^2 - 9x + 8 \right.}$$

$$= \frac{4}{3}\int_0^6 \left[x^2 - 12x + 18 \right] dx$$

$$= \frac{4}{3}\left[\int x^2 dx - 12\int x\,dx + \int 18\,dx \right]_0^6$$

$$= \frac{4}{3}\left[\frac{x^3}{3} - \frac{12x^2}{2} + 18x \right]_0^6$$

$$= \frac{4}{3}\left[\frac{6\times6\times\cancel{6}^{2}}{\cancel{3}} - \frac{\cancel{12}^{6}\times6\times6}{\cancel{2}} + 18\times6 \right]$$

$$= \frac{4}{3}\left[72 - 216 + 108 \right]$$

$$= \frac{4}{3}\left[180 - 216 \right] = \frac{4}{3} \times -\cancel{36}^{12}$$

$$= -48 \}$$

Green's theorem

$$\oint_C \{F_1 dx + F_2 dy\} = \iint_C \left\{\frac{\partial F_2}{dx} - \frac{\partial F_1}{\partial y}\right\} dx\,dy$$

~~Evaluate~~ $\int_C \{F_1 dx + F_2 dy\} = \iint \left\{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right\} dx\,dy$

Evaluate $\int_C \{\cos y\, \vec{i} + x(1-\sin y)\,\vec{j}\}\cdot d\vec{r}$ for a closed curve given by $x^2+y^2=1,\ z=0$ using Green's theorem.

Here given that $F_1 = \cos y$ and $F_2 = x(1-\sin y)$

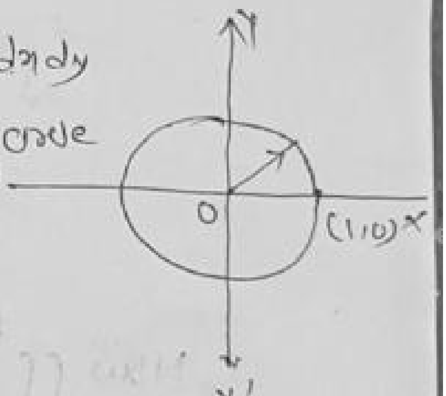
$\frac{\partial F_1}{\partial y} = -\sin y$, $\frac{\partial F_2}{\partial x} = 1(1-\sin y)$

By Green's theorem $I = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$

$= \iint_R \{1-\sin y - (\cancel{}-\sin y\}\,dx\,dy$

$= \iint_R \{1-\sin y + \sin y\}\,dx\,dy$

where R is the region area of curve $x^2+y^2=1$



$= \iint_R dx\,dy$

$= \pi \cdot 1^2 = \pi$

Verify Green's theorem in the plane for $\oint_C (xy+y^2)dx + x^2 dy$ where C is the closed curve of the region bounded by $x=y$ and $y=x^2$

Solution:- let $F_1 = xy+y^2$ and $F_2 = x^2$

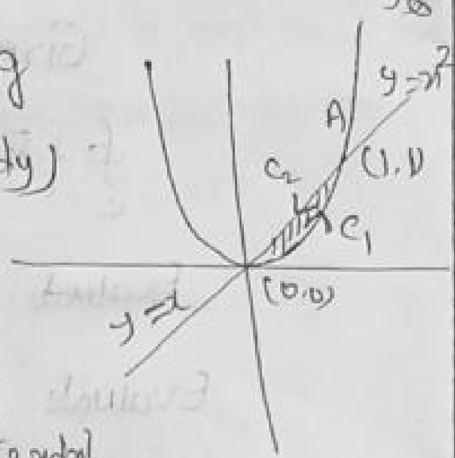
$\frac{\partial F_1}{\partial y} = x+2y$ $\qquad \frac{\partial F_2}{\partial x} = 2x$

let R be region bounded by the C. Along C_1, $y = x^2$

$dy = 2x\,dx$ the limits of x are 0 to 1

Line of Integral along

$C_1 = \int_{C_1} (F_1 dx + F_2 dy)$



$= \int_0^1 \{(xy + y^2) dx + x^2 dy\}$

$= \int_0^1 \{[x \cdot x^2 + x^4] dx + x^2 \cdot 2x dx\}$

$= \int_0^1 \{x^3 + x^4 + 2x^3\} dx = \int_0^1 (3x^3 + x^4) dx$

$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right] = \frac{3}{4} + \frac{1}{5} = \frac{15+4}{20} = \frac{19}{20}$

Along C_2, $y = x$ from point A to O

$dy = dx$ and the limit 1 to 0

Line integral $C_2 = \int_C (F_1 dx + F_2 dy)$

$\int_1^0 \{(xy + y^2) dx + x^2 dy\}$

$= \int_1^0 \{(x \cdot x + x^2) dx + x^2 dx\} = \int_1^0 \{3x^2 dx\}$

$-\left[3 \frac{x^3}{3}\right]_0^1 = -[1 - 0] = -1$

Line integral along $c = \frac{19}{25} - 1 = \frac{19-20}{20} = \frac{-1}{20}$

Now $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx dy = \iint_R (2x - x - 2y) dx dy$

$= \iint_R (x - 2y) dx dy$

$= \int_0^1 \left\{\int_{x^2}^{x} (x - 2y) dy\right\} dx$

$= \int_0^1 \left\{xy - \frac{\cancel{2}y^2}{\cancel{2}}\right\}_{x^2}^{x} dx$

$\int_0^1 \{(x \cdot x^2 - x^4) - (x \cdot x - x^2)\} dx$

$\int_0^1 (x^3 - x^4) dx$

$\int^x x^3 dx -$

$\int \{(x \cdot x - x^2) - (x \cdot x^2 - x^4)\} dx$

$\int x^4 dx - \int x^3 dx$

$\left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$

$\frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$

$\therefore \iint (F_1 dx + F_2 dy) = \iint \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx dy$

Volume Integral

let $\vec{F}$ be a vector point function and Volume V enclosed by a closed surface

The volume integral $= \iiint_V \vec{F}\, dv$

Q:→ If $\vec{F} = 2z\vec{i} - x\vec{j} + y\vec{k}$, evaluate $\iiint_V \vec{F}\, dv$ where V is the region bounded by the surfaces

$x=0$, $y=0$, $z=0$; $x=2$, $y=4$

$z=x^2$; $z=2$

<u>Solution</u> We know that the volume integral

$$\iiint_V \vec{F}\, dv = \iiint_V (2z\vec{i} - x\vec{j} + y\vec{k})\, dx\,dy\,dz$$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 \{2z\vec{i} - x\vec{j} + y\vec{k}\}\, dz$$

$$\cancel{= \int_0^2 dx \int_0^4 dy \left\{\frac{2z^2}{2}\vec{i} - xz\vec{j} + yz\vec{k}\right\}_{x^2}^2}$$

$$\cancel{= \int_0^2 dx \int_0^4 dx \{4\vec{i}}$$

$$= \int_0^2 dx \int_0^4 dy \left\{\frac{2z^2}{2}\vec{i} - xz\vec{j} + yz\vec{k}\right\}_{x^2}^2$$

$$= \int_0^2 dx \int_0^4 dy \{4\vec{i} - 2x\vec{j} + 2y\vec{k} - x^4\vec{i} + x^3\vec{j} - yx^2\vec{k}\}$$

$$= \int_0^2 dx \left[4y\vec{i} - 2xy\vec{j} + \frac{2y^2}{2}\vec{k} - x^4y\vec{i} + x^3y\vec{j} - \frac{y^2}{2}x^2\vec{k}\right]_0^4$$

$$= \int_0^2 dx \left[16\vec{i} - 8x\vec{j} + 16\vec{k} - 4x^4\vec{i} + 4x^3\vec{j} - 8x^2\vec{k}\right]$$

$$= \left[16x\vec{i} - 8x\vec{j} + 16x\vec{k} - \frac{4x^5}{5}\vec{i} + x^4\vec{j} - \frac{8x^3}{3}\vec{k}\right]_0^2$$

$$= \left[32\vec{i} - \cancel{16\vec{j}} + 32\vec{k} - \frac{128}{5}\vec{i} + \cancel{16\vec{j}} - \frac{64}{3}\vec{k}\right]$$

$$32\vec{i} - \frac{128}{5}\vec{i} + 32\vec{k} - \frac{64}{3}\vec{k} = \frac{32\vec{i}}{5} + \frac{32\vec{k}}{3}$$

Evaluate $\iiint_V \varphi\, dv$, where $\varphi = 45x^2y$ and V is the ~~closed~~ volume closed region bounded by the plane $4x+2y+z=8$, $x=0$, $y=0$, $z=0$

<u>Solution</u> $\iiint_V \varphi\, dv = \iiint_V 45x^2y\, dx\, dy\, dz$

$$= \int_0^2 \int_0^{4-2x} \int_0^{8-4x-2y} 45x^2y\, dx\, dy\, dz$$

$$= \int_0^2 dx \int_0^{4-2x} dy \left[45x^2y \int dz\right]_0^{8-4x-2y}$$

$$= \int_0^2 dx \int_0^{4-2x} dy \left[45x^2y\,(z)_0^{8-4x-2y}\right]$$

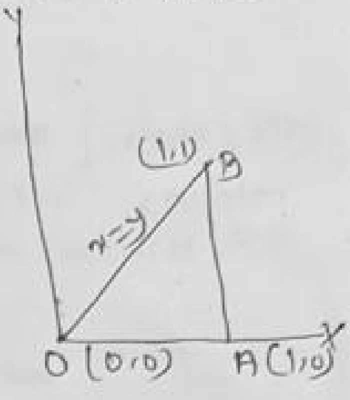
$$= \int_0^2 dx \int_0^{4-2x} dy \left[45x^2y\,(8-4x-2y)\right]$$

$$= 45\int_0^2 dx \int_0^{4-2x} (8x^2y - 4x^3y - 2x^2y^2)\, dy$$

$$= 45\int_0^2 dx \left[\frac{8x^2y^2}{2} - \frac{4x^3y^2}{2} - \frac{2x^2y^3}{3}\right]_0^{4-2x}$$

$$= 45\int_0^2 \left\{4x^2(4-2x)^2 - 2x^3(4-2x)^2 - \frac{2x^2}{3}(4-2x)^3\right\} dx$$

Using Green's theorem, evaluate $\iint (x^2y\,dx + x^2dy)$ where c is the boundary described counter of the triangle with vertices $(0,0), (0,1), (1,1)$



Here $F_1 = x^2y$ and $F_2 = x^2$

$$\frac{\partial F_1}{\partial y} = x^2, \quad \frac{\partial F_2}{\partial x} = 2x$$

$$= \int_0^1\int_0^x \left\{ \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial x} \right\} dx\,dy$$

$$= \int_0^1\int_0^x \left\{ 2x - x^2 \right\} dx\,dy$$

$$= \int_0^1 \int_0^1 \left\{ \int \right.$$

$$\int_0^1 (2x - x^2)\,dx \int_0^x dy = \int_0^1 (2x - x^2)\,dx\,[y]_0^x$$

$$\int_0^1 (2x - x^2)\,dx\,(x - 0) = \int_0^1 (2x^2 - x^3)\,dx$$

$$\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12} \text{ Ans}$$

Apply Green's theorem to evaluate $\oint_c [(y - \sin x)dx + \cos x\,dy]$, where c is the plane triangle enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$ and $y = \frac{2x}{\pi}$

$F_1 = y - \sin x, \quad F_2 = \cos x$

$$\frac{\partial F_1}{\partial y} = 1 - 0 \qquad \frac{\partial F_2}{\partial x} = -\sin x$$

By Green's theorem $\oint_C (F_1dx + F_2dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$

$$= \iint_R (-\sin x - 1)\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^{\frac{2x}{\pi}} (-\sin x - 1)\,dx\,dy$$

$$= -\int_0^{\pi/2}\left[y\sin x + y \right]_0^{\frac{2x}{\pi}} dy\,dx$$

$$= -\int_0^{\pi/2} \left\{ \frac{2x}{\pi}\sin x + \frac{2x}{\pi} - 0 \right\} dx$$

$$-\frac{2}{\pi}\int_0^{\pi/2}\{x\sin x + x\}\,dx$$

$$= -\frac{2}{\pi}\left\{-x\cos x - \int 1\cdot(-\cos x)\,dx + \frac{x^2}{2}\right\}_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left\{-\frac{\pi}{2}\cdot\cos\frac{\pi}{2} + \int \cos x\,dx + \frac{x^2}{2}\right\}_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left\{-\frac{\pi}{2}\times 0 + \sin x + \frac{x^2}{2}\right\}_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left\{0 + \sin\frac{\pi}{2} + \frac{\pi^2}{8}\right\} =$$

$$-\frac{2}{\pi}\times\frac{\pi^2}{8} = -\frac{2\pi}{8\,4} = -\frac{\pi}{4}$$

Apply Green's theorem to evaluate $\oint[(y-\sin x)dx + \cos x\,dy]$ where C is the plane triangle enclosed by the lines $y=0$, $x=\frac{\pi}{2}$ and $y = \frac{2x}{\pi}$

let $F_1 = y - \sin x$ and $F_2 = \cos x$

$\frac{\partial F_1}{\partial y} = 1$ and $\frac{\partial F_2}{\partial x} = -\sin x$

By Green's theorem

$$\oint_C (F_1\,dx + F_2\,dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy$$

$$= \iint_R(-\sin x - 1)\,dx\,dy = \int_0^{\pi/2}\int_0^{\frac{2x}{\pi}}(-\sin x - 1)\,dy\,dx$$

$$= \int_0^{\pi/2}\left[-y\sin x - y\right]_0^{\frac{2x}{\pi}}dx$$

$$= \int_0^{\pi/2}\left[-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right]dx$$

$$= -\frac{2}{\pi}\int_0^{\pi/2}[x\sin x + x]\,dx$$

$$= -\frac{2}{\pi}\left[-x\cos x + \int 1\cdot\cos x\,dx + \int x\,dx\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[-x\cos x + \sin x + \frac{x^2}{2}\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[\left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + \frac{\pi^2}{8}\right) - 0\right]$$

$$= -\frac{2}{\pi}\left[0 + 1 + \frac{\pi^2}{8}\right] = -\frac{2}{\pi} + -\frac{2}{\pi}\cdot\frac{\pi^{\cancel{2}}}{\cancel{8}\,4} = -\frac{2}{\pi} - \frac{\pi}{4}$$

$$= -\left(\frac{2}{\pi} + \frac{\pi}{4}\right)$$

Using Green's theorem, evaluate $\int_C (x^2y\,dx + x^2dy)$ where C is the boundary described counter clockwise of the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$

$\int$ By Green's theorem, we have

$$\int_C (F_1dx + F_2dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy$$

$F_1 = x^2y$ and $F_2 = x^2$

$\frac{\partial F_1}{\partial y} = x^2$ and $\frac{\partial F_2}{\partial x} = 2x$

$$= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy$$

$$= \int_0^1\int_0^x (2x - x^2)\,dx\,dy = \int_0^1 (2x - x^2)dx \int_0^x dy$$

$$= \int_0^1 [2x - x^2]dx\,[y]_0^x = \int_0^1 [2x - x^2]\,x\,dx$$

$$\int_0^1 (2x^2 - x^3)dx = \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$\frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12} \text{ Ans}$$

Apply Green's theorem to evaluate $\int_C [(2x^2-y^2)dx + (x^2+y^2)dy]$ where C is the boundary of the area enclosed by the x-axis and the upper half of circle $x^2+y^2=a^2$ 3.

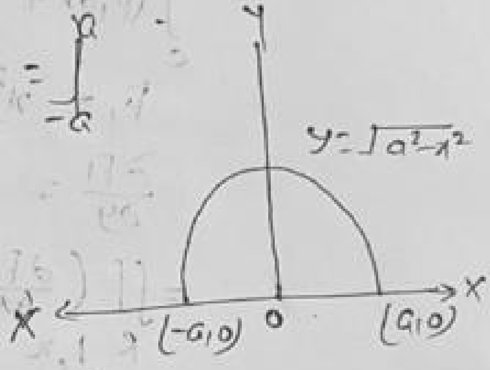
Here $F_1 = 2x^2 - y^2$ and $F_2 = x^2+y^2$

$\frac{\partial F_1}{\partial y} = -2y$ and $\frac{\partial F_2}{\partial x} = 2x$

By Green's theorem $\int_C (F_1 dx + F_2 dy) =$

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \iint_S (2x + 2y)\,dx\,dy = \int_{-a}^{a}$$



$$= \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} (2x+2y)\,dx\,dy$$

$$= 2\times 2\int_0^a dx \int_0^{\sqrt{a^2-x^2}} (x+y)\,dy$$

$$= 4\int_0^a dx \left[xy + \frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}}$$

$$= 4\int_0^a dx \left[x\sqrt{a^2-x^2} + \frac{a^2-x^2}{2} - 0\right]$$

$$= 4\left[\int_0^a x\sqrt{a^2-x^2}\,dx + \int_0^a \frac{a^2-x^2}{2}\,dx\right]$$

let $\sqrt{a^2-x^2} = p$

$a^2 - x^2 = p^2$

$-\cancel{2}x\,dx = \cancel{2}p\,dp$

$$-4\int_0^a p\cdot p\,dp + 4\left[\frac{1}{2}\int a^2 dx - \frac{\int x^2 dx}{2}\right]$$

$$= -4\left[\frac{p^3}{3}\right]_0^a + 4\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$-\frac{4}{3}\left[\sqrt{a^2-x^2}\right]^{\frac{3}{2}} + 4\left[a^3 - \frac{a^3}{3}\right]$$

$$-\frac{4}{3}\left[(a^2-a^2)\right]^{\frac{3}{2}} + 4\left(\frac{3a^3 - a^3}{3}\right)$$

$$0 + \frac{4\times 2a^3}{3} = \frac{8a^3}{3}$$

If $\vec{F} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$ evaluate the line integral $\int \vec{F}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the path c
$x = t,\ y = t^2,\ z = t^3$

Solution: At the point $(0,0,0)$

$x=0,\ y=0,\ z=0$
$t=0,\ t^2=0,\ t^3=0$

$\therefore x=t,\ y=t^2,\ z=t^3$

$dx = dt,\ dy = 2t\,dt,\ dz = 3t^2dt$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

Now $\int \vec{F}\cdot d\vec{r} = \int \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)$

$= [(3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}]\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$= [(3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}]\cdot(dt\hat{i} + 2t\,dt\hat{j} + 3t^2dt\hat{k})$

$= (3x^2+6y)dt - 14yz \times 2t\,dt + 20xz^2 \times 3t^2dt$

$= \int[(3t^2+6t^2)dt - 14t^2.t^3.2t\,dt + 20t.t^6 \times 3t^2dt]$

$= \int[(9t^2) - 28t^6 + 60t^9]dt$

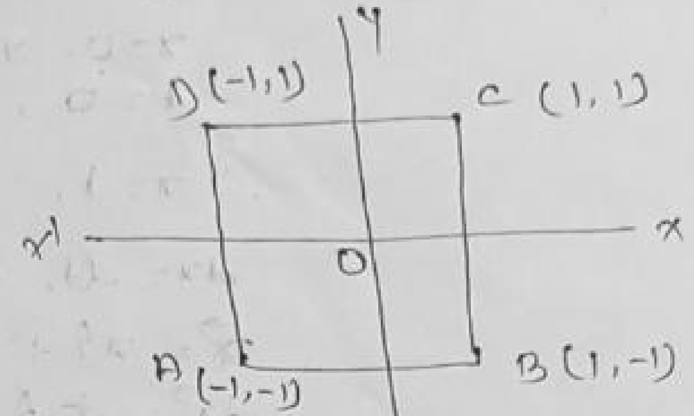
$\int_0^1 9t^2dt - 28\int_0^1 t^6dt + 60\int_0^1 t^9dt$

$\left[\frac{9t^3}{3}\right]_0^1 - 28\left[\frac{t^7}{7}\right]_0^1 + 60\left[\frac{t^{10}}{10}\right]_0^1$

$\frac{9}{3} - \frac{28}{7} + \frac{60}{10}$

$3 - 4 + 6 = 9 - 4 = 5$

Evaluate the line integral $\int_C [(x^2+xy)dx + (x^2+y^2)dy]$, where C is the square formed by the lines $y=\pm 1$ and $x=\pm 1$



from the figure of square ABCD

for line AB $y=-1$ $\therefore dy=0$ and $x=-1$ to 1

for line BC $x=1$ $\therefore dx=0$ and $y=-1$ to 1

for line CD $y=1$ $\therefore dy=0$ and $x=1$ to -1

for line DA $x=-1$ $\therefore dx=0$ and $y=1$ to -1

Here line integral $\int_C \{(x^2+xy)dx + (x^2+y^2)dy\}$

$$= \int_{AB}\{(x^2+x)dx + (x^2+y^2)dy\} + \int_{BC}\{(x^2+x)dx + (x^2+y^2)dy\} + \int_{CD}\{(x^2+xy)dx + (x^2+y^2)dy\} + \int_{DA}\{(x^2+xy)dx + (x^2+y^2)dy\}$$

$$= \int_{-1}^{1}(x^2+x)dx + \int_{-1}^{1}(1+y^2)dy + \int_{1}^{-1}(x^2+x)dx + \int_{1}^{-1}(1+y^2)dy$$

$$= \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{-1}^{1} + \left[y+\frac{y^3}{3}\right]_{-1}^{1} + \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{1}^{-1} + \left[y+\frac{y^3}{3}\right]_{1}^{-1}$$

$$= \left[\left(\frac{1}{3}+\frac{1}{2}\right)-\left(-\frac{1}{3}+\frac{1}{2}\right)\right] + \left[\left(1+\frac{1}{3}\right)-\left(-1-\frac{1}{3}\right)\right] + \left[\left(-\frac{1}{3}+\frac{1}{2}\right) - \left(\frac{1}{3}+\frac{1}{2}\right)\right]$$

$$= \left[\frac{5}{6} - \left(\frac{1}{6}\right)\right] + \left[\frac{4}{3} + \frac{4}{3}\right] + \left[\frac{1}{6} - \frac{5}{6}\right]$$

$$= \left[\frac{5-1}{6}\right] + \frac{8}{3} + \left[\frac{1-5}{6}\right]$$

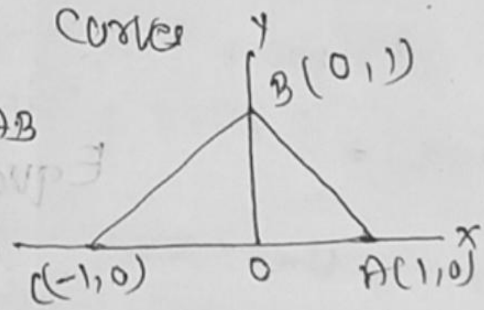
$$\cancel{\frac{4}{6}} + \frac{8}{3} - \cancel{\frac{4}{6}} = \frac{8}{3}$$

$$= \cancel{\int_{-1}^{1} (x^2+x)\,dx} + \int_{-1}^{1} (1+y^2)\,dy - \cancel{\int_{-1}^{1} (x^2+x)\,dx} - \int_{-1}^{1} (1+y^2)\,dy$$

$$= 0$$

Evaluate the line integral $\int_C (y^2 dx - x^2 dy)$ about the triangle whose vertices are $(1,0)$, $(0,1)$ and $(-1,0)$

<u>solution</u> let $A(1,0)$, $B(0,1)$ and $C(-1,0)$ be the points of curve



line ~~AB~~ integral along AB

Equation of AB is

$$\frac{x}{1} + \frac{y}{1} = 1$$

$$x+y=1 \quad \therefore y = 1-x$$

$$dy + dx = 0$$

$$\therefore dx = -dy, \; dy = -dx$$

$$\int_{AB} y^2 dx - x^2 dy = \int_{1}^{0} (1-x)^2 dx - x^2 \cdot dx$$

$$-\int_0^1 \{(1-x)^2 dx + x^2 dx\}$$

$$= -\int_0^1 \left(1-2x+x^2+x^2\right) dx = -\int_0^1 \left(1-2x+2x^2\right) dx$$

$$-\left\{ \int dx - 2\int x\,dx + 2\int x^2 dx \right\}_0^1$$

$$-\left\{x - \frac{2x^2}{2} + \frac{2x^3}{3}\right\}_0^1$$

$$= -\left\{\left(1 - 1 + \frac{2}{3}\right) - 0\right\} = -\frac{2}{3}$$

Equation of BC is $\frac{x}{-1} + \frac{y}{1} = 1$

$$-x + y = 1$$

$$y = 1 + x$$

$$dy = dx$$

$$\int_{BC} (y^2 dx - x^2 dy)$$

$$\int_0^0 \left\{(1+x)^2 dx - x^2 dx\right\}$$

$$= -\int_0^{-1} (1 + 2x + x^2 - x^2)\,dx$$

$$-\int_{-1}^{0} \left\{\int dx + 2\int x\,dx\right\}^0$$

$$= -\left\{x + \frac{2x^2}{2}\right\}_{-1}^{0}$$

$$= -\left\{0 + 0 - 1 + 1\right\} = 0$$

Equation of CA is $y = 0$ $\therefore dy = 0$

$$x + y = 1$$

$$dx + dy = 0$$

$$dx + 0 = 0$$

$$dx = 0$$

$$\int_{-1}^{1} y^2 dx - x^2 dy = \int_{-1}^{1} (y \times 0 - x^2 \times 0) = 0$$

area of $\triangle ABC$ $\int_C (y^2 dx - x^2 dy) =$

$$\int_{AB} y^2 dx - x^2 dy + \int_{BC} y^2 dx - x^2 dy + \int_{CA} (y^2 dx - x^2 dy)$$

$$= -\frac{2}{3} + 0 + 0 = -\frac{2}{3}$$

Gauss Divergence theorem
OR
Relation between Surface Integral and Volume Integral.

Mathematically,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \mathrm{div}\,\vec{F}\,dv$$

$$\nabla = \mathrm{div} = \phi = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

Find $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = (2x+3z)\vec{i} - (xz+y)\vec{j} + (y^2+2z)\vec{k}$ and S is the surface of the sphere having centre at $(3,-1,2)$ and radius 3.

<u>Solution</u>. let V be the volume enclosed by the Surface S.

We know that Gauss divergence theorem

$$\iint_S \vec{F}\cdot n\,ds = \iiint_V \mathrm{Div}\,\vec{F}\,dv$$

$$= \iiint_V \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left\{(2x+3z)\vec{i} + (-xz-y)\vec{j} + (y^2+2z)\vec{k}\right\} dv$$

$$= \iiint_V \left\{\frac{\partial}{\partial x}(2x+3z) + \frac{\partial}{\partial y}(-xz-y) + \frac{\partial}{\partial z}(y^2+2z)\right\} dv$$

$$= \iiint_V (2-1+2)\,dv$$

$$= \iiint_V 3\,dv = 3\iiint_V dv$$

~~Volume~~ V is the volume of the sphere of radius 3

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}$$

$$= 3\times\frac{4}{3}\times\pi\times3\times3\times3 = 108\pi$$

Q → The vector field $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ is defined over the volume of the cuboid given by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ enclosing the surface S. Evaluate the surface integral $\iint_S \vec{F}\cdot\hat{n}\,ds$

Solution By divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

$$= \iiint_V \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2\vec{i} + z\vec{j} + yz\vec{k})\,dv$$

$$= \iiint_V \left(\frac{d}{dx}x^2 + \frac{dz}{dy} + \frac{d}{dz}yz\right)dv$$

$$= \iiint (2x + 0 + y)\,dv$$

$$= \int_0^a\int_0^b\int_0^c (2x + y)\,dx\,dy\,dz$$

$$= \int_0^a dx\int_0^b dy\int_0^c (2x + y)\,dz$$

$$= \int_0^a dx\int_0^b dy\left\{2xz + yz\right\}_0^c$$

$$= \int_0^a dx\int_0^b dy\,[2xc + yc]$$

$$= \int_0^a dx\left[\int (2xc + yc)\,dy\right]_0^b$$

$$= \int_0^a dx\left[2xyc + \frac{y^2}{2}c\right]_0^b$$

$$= \int_0^a dx\left[2xbc + \frac{b^2}{2}c\right]$$

$$= \int_0^a \left(2xbc + \frac{b^2}{2}c\right)dx$$

$$= \left[\frac{\cancel{2}x^2}{\cancel{2}}bc + \frac{xb^2c}{2}\right]_0^a \qquad abc\left(a + \frac{b}{2}\right) \text{ Ans}$$

$$a^2bc + \frac{ab^2c}{2}$$

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Use Gauss divergence theorem for $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}$ taken over the ~~regular~~ rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b$, $0 \le z \le c$

We know that Gauss divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\vec{F}\,.dv$$

$$= \iiint_V \left\{\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right\}\left\{(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right\} dv$$

$$\iiint_V \left[\frac{d}{dx}(x^2-yz) + \frac{d}{dy}(y^2-zx) + \frac{d}{dz}(z^2-xy)\right] dv$$

$$= \cancel{2x+2y+2z)}$$

$$\iiint_V \left[(2x+2y+2z)\right] dv$$

$$= 2\int_0^a\int_0^b\int_0^c (x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^a dx\int_0^b dy\int_0^c (x+y+z)\,dz$$

$$= 2\int_0^a dx\int_0^b dy\left[xz+yz+\frac{z^2}{2}\right]_0^c$$

$$= 2\int_0^a dx\int_0^b dy\left[xc+yc+\frac{c^2}{2}\right]$$

$$= 2\int_0^a dx\left[\int\left(xc+yc+\frac{c^2}{2}\right)dy\right]_0^b$$

$$= 2\int_0^a dx\left[xyc+\frac{y^2}{2}+\frac{c^2}{2}y\right]_0^b$$

$$= 2\int_0^a dx\left[xbc+\frac{b^2c}{2}+\frac{c^2b}{2}\right]$$

$$= 2\int\left[xbc+\frac{1}{2}b^2c+\frac{c^2b}{2}\right]_0^a dx$$

$$= 2\left[\frac{x^2}{2}bc+\frac{b^2xc}{2}+\frac{xc^2b}{2}\right]$$

$$= 2\left[\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}\right]$$

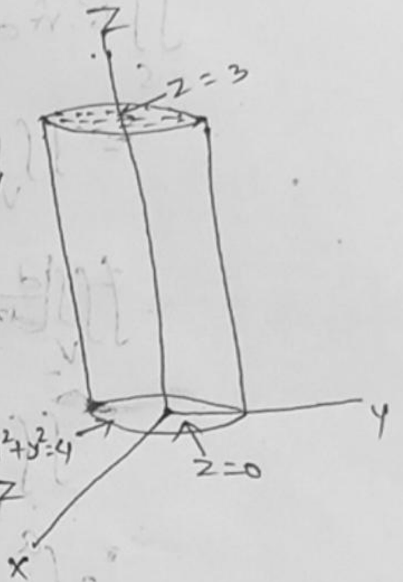
$$= \cancel{2}\,abc\,\frac{(a+b+c)}{\cancel{2}}$$

$$= abc(a+b+c)$$

Evaluate $\iint_S \vec{F}.ds$, where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounding the region $x^2+y^2=4$, $z=0$ and $z=3$

Solution: We know that by divergence theorem

$$\iint_S \vec{F}\,\hat{n}\,ds = \int_V \operatorname{div}\vec{F}\,dV$$



$$\int_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}z^2\right]dV$$

$$= \iiint_V \left[4 - 4y + 2z\right]dV$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4-4y+2z)\,dz\,dy\,dz$$

$$= 4\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^3 \left\{4z - 4yz + \cancel{2}\frac{z^2}{\cancel{2}}\right]_0^3 dy\,dx$$

$$= 4\int_0^2\int_0^{\sqrt{4-x^2}} \left(12 - 12y + 9\right]dy\,dx$$

$$= 4\int_0^2 dx\left\{\int_0^{\sqrt{4-x^2}} 21 - 12y\right\}dy = 4\int_0^2 dx\left[21y - \frac{12y^2}{2}\right]_0^{\sqrt{4-x^2}}$$

$$= 4\left[\int_0^2 dx\left[21\sqrt{4-x^2} - 6(4-x^2)\right)\right.$$

$$4\times\left[\int_0^2 21\sqrt{4-x^2} - 24 + 6x^2\right]dx$$

$$4\left[21\int\sqrt{4-x^2} - 24\int dx + 6\int x^2 dx\right]$$

$$84\int\sqrt{4-x^2}\,dx - \left[24x\right]_0^2 + \left[6x\frac{x^3}{3}\right]_0^2$$

$$= 84\left\{\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - 0 + 0$$

$$84\left[\frac{2\sqrt{4-4}}{2} + \frac{4}{2}\sin^{-1}\frac{2}{2}\right] = 84\times\cancel{2}\times\frac{\pi}{\cancel{2}}$$

$$84\pi$$

Stoke's theorem
Relation between surface and line integrals
Here $\vec{F}$ is a vector field function and C is a simple closed curved surface integral

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot \hat{n}\, ds$$

$$ds = \int \frac{dx\,dy}{|\hat{n}\,\hat{k}|}$$

Use stoke's theorem, evaluate

$$\int_C \left[(x+y)\,dx + (2x-z)\,dy + (y+z)\,dz\right]$$ where c is the boundary of the triangle with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$

Here $\vec{F} = (x+y)\vec{i} + (2x-z)\vec{j} + (y+z)\vec{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(y+z) - \frac{\partial}{\partial z}(2x-z)\right] - \vec{j}\left[\frac{\partial}{\partial x}(y+z) - \frac{\partial}{\partial z}(x+y)\right] + \vec{k}\left[\frac{\partial}{\partial x}(2x-z) - \frac{\partial}{\partial y}(x+y)\right]$$

$$= i\,[1+1] - J[0-0] + \vec{k}(2-1) = 2\vec{i} - 0 + \vec{k} = 2\vec{i}+\vec{k}$$

We find out equation of the plane passing through the point $A(2,0,0)$, $B(0,3,0)$ and $C(0,0,6)$

$$\frac{x}{A} + \frac{y}{B} + \frac{z}{C} = 1$$

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$$

$$3x + 2y + z = 6$$

Vector $\hat{n}$ normal to this plane is

$$\nabla(3x + 2y + z - 6) = 3\vec{i} + 2\vec{j} + \vec{k}$$

$$\hat{n} = \frac{3\vec{i} + 2\vec{j} + \vec{z}}{\sqrt{9+4+1}} = \frac{3\vec{i} + 2\vec{j} + \vec{z}}{\sqrt{14}}$$

$$\oint_C [(x+y)dx + (2x-y)dy + (y+z)dz] =$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \hat{n}\, ds$$

$$= \iint (2\vec{i} + \vec{k}) \cdot \frac{(3\vec{i} + 2\vec{j} + \vec{k})}{\sqrt{14}}\, ds$$

$$\frac{6+1}{\sqrt{14}}\, ds = \frac{7}{\sqrt{14}}\, ds$$

$\frac{7}{\sqrt{14}}$ Area of ΔABC

area of ΔABC so change ~~into~~ ds to $\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$ where $\iint dx\,dy$ is area of ΔAOB

$$\frac{7}{\sqrt{14}} \iint_S \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|} = \frac{7}{\sqrt{14}}\ \hat{n} \to 3\vec{i} + 2\vec{j}$$

$$\frac{7}{\sqrt{14}} \times \frac{dx\,dy}{\frac{(3\vec{i} + 2\vec{j} + \vec{k})\cdot\vec{k}}{\sqrt{14}}}$$

$$\Rightarrow \frac{7}{\sqrt{14}} \times \frac{dx\,dy}{\frac{1}{\sqrt{14}}} = 7 \iint dx\,dy$$

$7(\text{area of } \Delta AOB) = 7 \times \frac{1}{2} \times \text{Base} \times h$

$7 \times \frac{1}{2} \times 2 \times 3 = 21$

Evaluate by stoke's theorem $\oint_C (e^x dx + 2y dy - dz)$ where c is the curve $x^2+y^2=4, z=2$

We know that by stoke's theorem

$$\oint_C \vec{F} d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds \qquad ds = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

$$\hat{n} = \frac{xi + yj + zk}{\sqrt{x^2+y^2+z^2}}$$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$d\vec{r} = \cancel{dx}\ \vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz$

$\vec{F} = e^x\vec{i} + 2y\vec{j} - \vec{k}$

$\oint_C \vec{F} d\vec{r} = \iint_S \cancel{\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)}(e^x i + 2y j - k)$

$\cancel{= e^x}$

$$\iint_S (\nabla \times \vec{F})\, \hat{n} \cdot ds$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-1) - \frac{\partial}{\partial z}(2y)\right] - \vec{j}\left[\frac{\partial}{\partial x}(-1) - \frac{\partial e^x}{\partial z}\right] + \vec{k}\left[\frac{\partial 2y}{\partial x} - \frac{\partial e^x}{\partial y}\right]$$

$$= i[0-0] - \vec{j}[0-0] + \vec{k}[0-0] = 0$$

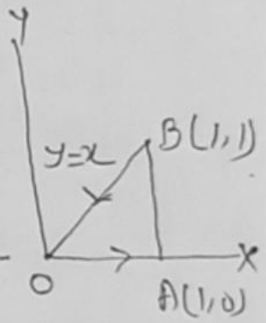
$$\iint_S (\text{curl} \times F)\, \hat{n}\, ds = 0 \times \hat{n} \cdot ds = 0$$

Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ by Stoke's theorem, where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$ and c is the boundary of the traingle with vertices at $(0,0,0)$, $(1,0,0)$ and $(1,1,0)$

Solution :- Since z-co-ordinates of each vertex of the traingle is zero, therefore, the traingle lies in the xy-plane and $\hat{n} = \hat{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = \hat{j} + 2(x-y)\hat{k}$$

$$\text{curl } \vec{F}\cdot\hat{n} = [\hat{j} + 2(x-y)\hat{k}]\cdot\hat{k}$$



The equation of the line OB is $y = x$

By Stoke's theorem $\oint \vec{F}\cdot d\vec{r} = \iint \text{curl } \vec{F}\cdot\hat{n}\,ds$

$$= \int_0^1 \int_0^x 2(x-y)\,dy\,dx$$

$$= \int_0^1 2\left[xy - \frac{y^2}{2}\right]_0^x dx$$

$$= 2\int_0^1 \left(x\cdot x - \frac{x^2}{2}\right) dx$$

$$= 2\left[\int_0^1 x^2 dx - \frac{1}{2}\int_0^1 x^2 dx\right]$$

$$= 2\left[\frac{x^3}{3} - \frac{x^3}{2\cdot 3}\right]_0^1$$

$$= 2\left[\frac{1}{3} - \frac{1}{6}\right] = 2\left[\frac{2-1}{6}\right]$$

$$\frac{\cancel{2}\times 1}{\cancel{6}3} = \frac{1}{3} \text{ Ans}$$